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Cycle Index Formulas for D_n Acting on Unordered Pairs

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Abstract

The cycle index of dihedral group D_n acting on the set X of the vertices of a regular n-gon was studied (See [1]). In this paper we study the cycle index formulas of D_n acting on unordered pairs from the set $X = \{1, 2, \ldots, n\}$. In each case the actions of the cyclic part and the reflection part are studied separately for both an even value of n and an odd value of n.

1. Introduction

The concept of the cycle index was discovered by Polya (See [2]) and he gave it its present name. He used the cycle index to count graphs and chemical compounds via the Polya's Enumeration Theorem. More current cycle index formulas include the cycle index of the reduced ordered triples groups $S_n^{[3]}$ (See [3]) which was further extended by Kamuti and Njuguna to cycle index of the reduced ordered r-group $S_n^{[r]}$ (See [4]). The Cycle Index of Internal Direct Product Groups was done in 2012 (See [5]).

2. Definitions and Preliminaries

Definition 1.
A cycle index is a polynomial in several variables which is structured in such a way that information about how a group of permutations acts on a set can be simply read off from the coefficients and exponents.

Definition 2.
A cycle type of a permutation is the data of how many cycles of each length are present in the cycle decomposition of the permutation.

Definition 3.
A monomial is a product of powers of variables with nonnegative integer exponents possibly with repetitions.

Preliminary result 1
Let (G, X) be a finite permutation group and let $X^{(2)}$ denote the set of all 2-element subsets of X. If g is a permutation in (G, X) we want to know the disjoint cycle structure of the permutation g' induced by g on $X^{(2)}$.
We shall briefly sketch the technique (we call it the pair group action) for obtaining the disjoint cycle structure of g'; for a detailed explanation and examples (See [6]).
Let $mon(g) = t_1^{\alpha_1} t_2^{\alpha_2} \ldots t_n^{\alpha_n}$, our aim is to find $mon(g')$. To do this there are two separate contributions from g to the corresponding term of $mon(g')$ which we need to consider:

(i) from pairs of points both lying in a common cycle of g.
(ii) from pairs of points each from a different cycle of g.

It is convenient to divide the first contribution into:

(a) Those pairs from odd cycles,
(b) Those pairs from even cycles.

(i) (a) If we let $\theta = (123\ldots 2m+1)$ be an odd cycle in g and we let the elements in a pair come from a common cycle, then the permutation θ' in $(G,X^{(2)})$ induced by θ is as follows:

$$\{123\ldots2m+1\}\rightarrow\left\{\begin{array}{l}\{1,2\}\rightarrow\{2,3\}\rightarrow\{3,4\}\rightarrow\ldots\rightarrow\{2m+1,1\}\\ \{1,3\}\rightarrow\{2,4\}\rightarrow\{3,5\}\rightarrow\ldots\rightarrow\{2m,1\}\rightarrow\{2m+1,2\}\\ \{1,4\}\rightarrow\{2,5\}\rightarrow\{3,6\}\rightarrow\ldots\rightarrow\{2m-1,1\}\rightarrow\{2m,2\}\rightarrow\{2m+1,3\}\\ \ldots\\ \{1,m\}\rightarrow\{2,m+1\}\rightarrow\{3,m+2\}\rightarrow\ldots\rightarrow\{m+3,1\}\rightarrow\ldots\rightarrow\{2m+1,m-1\}\\ \{1,m+1\}\rightarrow\{2,m+2,\}\rightarrow\ldots\rightarrow\{m+2,1\}\rightarrow\ldots\rightarrow\{2m+1,m\}\end{array}\right.$$

Hence $t_{2m+1} \longrightarrow b_{2m+1}^{m}$.

So if we have α_{2m+1} cycles of length $2m+1$ in g, the pairs of points lying in the common cycles contribute;

$$t_{2m+1}^{\alpha_{2m+1}} \longrightarrow b_{2m+1}^{m(\alpha_{2m+1})} \qquad (2.1) \qquad \text{for odd cycles.}$$

(i) (b) If we let $\theta = (123\ldots 2m)$ be an even cycle in g and we let the elements in a pair come from a common cycle, then the permutation θ' in $(G,X^{(2)})$ induced by θ is as follows:

$$\{123\ldots2m\}\rightarrow\left\{\begin{array}{l}\{1,2\}\rightarrow\{2,3\}\rightarrow\{3,4\}\rightarrow\ldots\rightarrow\{2m,1\}\\ \{1,3\}\rightarrow\{2,4\}\rightarrow\{3,5\}\rightarrow\ldots\rightarrow\{2m-1,1\}\rightarrow\{2m,2\}\\ \ldots\\ \{1,m-1\}\rightarrow\{2,m\}\rightarrow\{3,m+1\}\rightarrow\ldots\rightarrow\{m+3,1\}\rightarrow\ldots\rightarrow\{2m,m-2\}\\ \{1,m\}\rightarrow\{2,m+1\}\rightarrow\{3,m+2\}\rightarrow\ldots\rightarrow\{m+2,1\}\rightarrow\ldots\rightarrow\{2m,m-1\}\\ \{1,m+1\}\rightarrow\{2,m+2\}\rightarrow\ldots\rightarrow\{2,m+2\}\rightarrow\ldots\rightarrow\{m,2m\}\end{array}\right.$$

Hence $t_{2m} \longrightarrow b_m b_{2m}^{m-1}$.

So if α_{2m} is the number of cycles of length $2m$ in g, the pairs of points lying in common cycle contribute:

$$t_{2m}^{\alpha_{2m}} \longrightarrow (b_m b_{2m}^{m-1})^{\alpha_{2m}} \qquad (2.2) \qquad \text{for even cycles.}$$

(ii) Consider two distinct cycles of length a and b in (G,X). If x belongs to an a-cycle θ_a of g and y belongs to a b-cycle θ_b of g, then the least positive integer β for which $g^{\beta}x = x$ and $g^{\beta}y = y$ is $[a,b]$, (the lcm of a and b).So the element (x,y) belongs to an $[a,b]$-cycle of g'.

The number of such $[a,b]$-cycles contributed by g on $\theta_a \times \theta_b$ to g' is the total number of pairs in $\theta_a \times \theta_b$ divided by $[a,b]$, the length of each cycle.

This number is therefore $\frac{ab}{[a,b]} = (a,b)$, the gcd of a and b.

In particular if $a = b = l$, the contribution by g on $\theta_a \times \theta_b$ to g' is l cycles of length l.
Thus when $a \neq b$ we have;

$$t_a^{\alpha_a} t_b^{\alpha_b} \longrightarrow b_{[a,b]}^{(a,b)\alpha_a\alpha_b} \qquad (2.3)$$

and when $a = b = l$

$$t_l^{\alpha_l} \longrightarrow b_l^{l\binom{\alpha_l}{2}} \qquad (2.4)$$

Preliminary result 2

The cycle index formulas of dihedral group D_n acting on the set X of the vertices of a regular n-gon are given by:

$$Z_{Dn,X} = \frac{1}{2n}\left[\sum_{d|n}\phi(d)\,t_d^{\frac{n}{d}} + \frac{n}{2}t_1^2 t_2^{\frac{n-2}{2}} + \frac{n}{2}t_2^{\frac{n}{2}}\right] \qquad 2.5(a)$$

if n is even and

$$Z_{Dn,X} = \frac{1}{2n}\left[\sum_{d|n}\phi(d)\,t_d^{\frac{n}{d}} + n t_1 t_2^{\frac{n-1}{2}}\right] \qquad 2.5(b)$$

if n is odd.

Where ϕ is the Euler's phi formula.
For the proof to these important results (See [7],[8]).

3. Cycle index of D_n acting on unordered pairs

With the help of the above important results we now study the cycle index of D_n acting on unordered pairs of the set $X = \{1,2,\dots,n\}$.

3.1 Case 1: if n is even

We first consider the cyclic part from 2.5(a) $Z_{C_n,X} = \frac{1}{n}\sum_{d|n}\phi(d)\,t_d^{\frac{n}{d}}$

Since n is even, then the divisors of n can either be even or odd.
If d is even and

(i) the pair come from a common cycle then from 2.2 we have;

$$t_d^{\frac{n}{d}} \longrightarrow \left(b_{\frac{d}{2}} b_d^{\frac{d}{2}-1}\right)^{\frac{n}{d}} \qquad (3.1.1)$$

(ii) each of the elements in the pair comes from a different cycle then from 2.4 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d\binom{\frac{n}{d}}{2}} \qquad (3.1.2)$$

Combining (3.1.1) and (3.1.2) we have for an even divisor

$$\left(b_{\frac{d}{2}} b_d^{\frac{d}{2}-1}\right)^{\frac{n}{d}} b_d^{d\binom{\frac{n}{d}}{2}} = b_{\frac{d}{2}}^{\frac{n}{d}} b_d^{\frac{n^2-2n}{2d}} \qquad (3.1.3)$$

If d is odd and;

(i) the pair come from a common cycle then from 2.1 we have:

$$t_d^{\frac{n}{d}} \longrightarrow \left(b_d^{\frac{d-1}{2}}\right)^{\frac{n}{d}} \qquad (3.1.4a)$$

(ii) each of the elements in the pair comes from a different cycle then from 2.4 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d\binom{\frac{n}{d}}{2}} \qquad (3.1.4b)$$

Combining (3.1.4a) and (3.1.4b) we have for an even divisor;

$$\left(b_d^{\frac{d-1}{2}}\right)^{\frac{n}{d}} b_d^{d\binom{\frac{n}{d}}{2}} = b_d^{\frac{n^2-n}{2d}} \qquad (3.1.5)$$

Therefore the cycle index formula for C_n acting on $X^{(2)}$ when n is even is given by:

$$\frac{1}{n}\left[\sum_{\substack{d|n\\2|d}} \phi(d)\left(b_{\frac{d}{2}}^{\frac{n}{d}} b_d^{\frac{n^2-2n}{2d}}\right) + \sum_{\substack{d|n\\2\nmid d}} \phi(d)\left(b_d^{\frac{n^2-n}{2d}}\right)\right] \qquad (3.1.6)$$

From 2.5(a) we note that the two different kinds of reflections (a reflection through two vertices and a reflection through the edges) induce different monomials when acting on the vertices of a regular n-gon for n even.
We now investigate the induced monomials when the reflections act on $X^{(2)}$

(i) We first consider the part $t_1^2 t_2^{\frac{n-2}{2}}$

If each of the elements in the pair come from a different cycle of a different length then from 2.3 we have;

$$t_1^2 t_2^{\frac{n-2}{2}} \longrightarrow b_2^{n-2} \qquad (3.1.7)$$

If both come from different cycles of same length, then there are two cases:
Either both come from cycles of length one in which from 2.4 we have;

$$t_1^2 \longrightarrow b_1 \qquad (3.1.8a).$$

Or each come from a different cycle of length two then from 2.4 we have;

$$t_2^{\frac{n-2}{2}} \longrightarrow b_2^{2\binom{\frac{n-2}{2}}{2}} \qquad (3.1.8b)$$

If both come from a common cycle of length two, then from 2.2 we have:

$$t_2^{\frac{n-2}{2}} \longrightarrow b_1^{\frac{n-2}{2}} \qquad (3.1.9)$$

Combining (3.1.7) (3.1.8a) (3.1.8b) and (3.1.9), we get;

$$b_1^{\frac{n-2}{2}+1} b_2^{2\binom{\frac{n-2}{2}}{2}+(n-2)} = b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}.$$

But from 2.5(a) we have $\frac{n}{2}$ monomials of the form $t_1^2 t_2^{\frac{n-2}{2}}$ and hence a total of

$$\frac{n}{2} b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}} \qquad (3.1.10)$$

monomials will be induced.

(ii) Next we consider the part $t_2^{\frac{n}{2}}$

If the pair comes from a common cycle then from 2.2 we have;

$$t_2^{\frac{n}{2}} \longrightarrow (b_1 b_2^0)^{\frac{n}{2}} = b_1^{\frac{n}{2}} \quad (3.1.11)$$

If the pair comes from different cycles then the cycles are of length 2 and from 2.4 we have;

$$t_2^{\frac{n}{2}} \longrightarrow b_2^{2\binom{n/2}{2}} \quad (3.1.12)$$

Combining (3.1.11) and (3.1.12) we have; $b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}$ but from 2.5(a) we have $\frac{n}{2}$ monomials of the form $t_2^{\frac{n}{2}}$

and hence a total of $\frac{n}{2} b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}$ (3.1.13)

monomials will be induced.

Adding (3.1.10) and (3.1.13) we have; $n\, b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}$ (3.1.14)

Now adding (3.1.6) and (3.1.14) we have the cycle index formula;

$$Z_{D_n, X^{(2)}} = \frac{1}{2n}\left[\sum_{\substack{d|n \\ 2|d}} \phi(d)\left(b_{\frac{d}{2}}^{\frac{n}{d}} b_d^{\frac{n^2-2n}{2d}}\right) + \sum_{\substack{d|n \\ 2\nmid d}} \phi(d)\left(b_d^{\frac{n^2-n}{2d}}\right) + n\, b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}\right] \quad (3.1.15)$$

3.2 Case 2: if n is odd

We first consider the cyclic part $Z_{C_n, X} = \frac{1}{n}\sum_{d|n} \phi(d) t_d^{\frac{n}{d}}$

In this case d must be odd since n is odd and an odd number is not divisible by even number.
If a pair comes from a common cycle, then from 2.1 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{\frac{n(d-1)}{2d}} \quad (3.2.1).$$

If the pair come from different cycles, of the same length then from 2.4 we have;

$$t_d^{\frac{n}{d}} \longrightarrow b_d^{d\binom{\frac{n}{d}}{2}} \quad (3.2.2).$$

Combining (3.2.1) and (3.2.2) we have $b_d^{\frac{n^2-nd}{2d}+\frac{n(d-1)}{2d}} = b_d^{\frac{n^2-n}{2d}}$ (3.2.3).

Therefore the cycle index formula of C_n acting on $X^{(2)}$ when n is odd is given by;

$$\frac{1}{n}\left[\sum_{d|n} \phi(d) b_d^{\frac{n^2-n}{2d}}\right] \quad (3.2.4)$$

For us to study the induced monomials by the reflection symmetries, it is important to note that all the reflection symmetries of a regular n-gon with n odd have their line of symmetry passing through a vertex and an edge.

We now consider the monomials induced by the reflection part $t_1 t_2^{\frac{n-1}{2}}$;

If both of the elements come from a common cycle then the cycle has to be of length two and hence from 2.2 we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow (b_1 b_2^0)^{\frac{n-1}{2}} = b_1^{\frac{n-1}{2}} \qquad (3.2.5).$$

If the elements in the pair come from different cycles with one from a cycle of length one and the other from a cycle of length two then from 2.3 we have;

$$t_1 t_2^{\frac{n-1}{2}} \longrightarrow b_2^{\frac{n-1}{2}} \qquad (3.2.6).$$

If a pair comes from different cycles of length two, then from 2.4 we have;

$$t_2^{\frac{n-1}{2}} \longrightarrow b_2^{2\binom{\frac{n-1}{2}}{2}} \qquad (3.2.7)$$

Combining (3.2.5), (3.2.6) and (3.2.7) we have;

$$b_1^{\frac{n-1}{2}} b_2^{\frac{n-1}{2}+\frac{n^2+3-4n}{4}} = b_1^{\frac{n-1}{2}} b_2^{\frac{n^2-2n+1}{4}}$$

We note that from 2.5(b) there are n monomials of the form $t_1 t_2^{\frac{n-1}{2}}$ and hence n monomials will be induced and hence we have;

$$nb_1^{\frac{n-1}{2}} b_2^{\frac{n-1}{2}+\frac{n^2+3-4n}{4}} = nb_1^{\frac{n-1}{2}} b_2^{\frac{n^2-2n+1}{4}} \qquad (3.2.8)$$

Adding (3.2.4) and (3.2.8) we have the cycle index formula as;

$$Z_{Dn,X^{(2)}} = \frac{1}{2n}\left[\sum_{d|n} \phi(d)\, b_d^{\frac{n^2-n}{2d}} + nb_1^{\frac{n-1}{2}} b_2^{\frac{n^2-2n+1}{4}}\right] \qquad (3.2.9)$$

Example 1

Let $n = 7$ then the dihedral group D_7 of degree 7 acting on the unordered pairs of the set $= \{1,2,3,4,5,6,7\}$. Then;

$|D_7| = 14$, $d = 1,7$, $\phi(1) = 1$ and $\phi(7) = 6$

Hence from 3.2.9 we have;

$$Z_{D_7,X^{(2)}} = \frac{1}{14}[b_1^{21} + 7b_1^3 b_2^9 + 6b_7^3] .$$

Example 2

Let $n = 6$, then the dihedral group D_6 of degree 6 acting on the unordered pairs of the set $X = \{1,2,3,4,5,6\}$. Then;

$|D_6| = 12$, $d = 1,2,3,6$, $\phi(1) = 1$, $\phi(2) = 1, \phi(3) = 2$ and $\phi(6) = 2$.

Hence from 3.1.15 we have;

$$Z_{D_6,X^{(2)}} = \frac{1}{12}[b_1^{15} + 2b_3^5 + 7b_1^3 b_2^6 + 2b_3 b_6^2].$$

4. Conclusion

The cycle index formulas of D_n acting on unordered pairs are given as;

$$Z_{Dn,X^{(2)}} = \frac{1}{2n}\left[\sum_{d|n} \phi(d)\, b_d^{\frac{n^2-n}{2d}} + n b_1^{\frac{n-1}{2}} b_2^{\frac{n^2-2n+1}{4}}\right]$$ for an odd value of *n* and

$$Z_{Dn,X^{(2)}} = \frac{1}{2n}\left[\sum_{\substack{d|n \\ 2|d}} \phi(d) \left(b_{\frac{d}{2}}^{\frac{n}{d}} b_d^{\frac{n^2-2n}{2d}}\right) + \sum_{\substack{d|n \\ 2\nmid d}} \phi(d) \left(b_d^{\frac{n^2-n}{2d}}\right) + n\, b_1^{\frac{n}{2}} b_2^{\frac{n^2-2n}{4}}\right]$$ for an even value of *n*.

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