MATHEMATICAL MODELLING AND SIMULATION OF COMPETITION FOR STUDENTS' POPULATION VIA INFLUENCE AND ECONOMIC FACTORS WITH HOLLING TYPE II RESPONSE

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DECLARATION

This research project is my original work and has not been presented elsewhere for a degree or any other award.

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DEDICATION

Special dedication to my beloved mum Anastasia Otieno who continually encouraged me in times of difficulty during my study period. I therefore dedicate this noble work to her as my special one.

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ABBREVIATIONS AND ACRONYMS

ARIMA	Auto Regressive Integrated Moving Averages
EFA	Education For All
iOS	IPhone Operating System
FPE	Free Primary Education
GA	Genetic Algorithm
L-V	Lotka-Volterra
MAPE	Mean Percentage Error
MIC	Market Intelligence and Consulting
NGO	Non-Governmental Organizations
NLS	Non-linear Least Square
ODE	Ordinary Differential Equations
SARIMA	Seasonal Auto Regressive Integrated Moving Averages
UNESCO	United Nations Educational, Scientific Cultural Organizations

OPERATIONAL DEFINITION OF TERMS

Amensalism	Dependence of two where one benefits from the other but not all of them.
Big four agenda	Policy area to which the Kenyan government dedicated vast resources during the 2018-2022 period namely, food security, affordable housing manufacturing and universal healthcare for accelerated socioeconomic transformation, increased job creation and improved quality of life
Holling type response	Model where one species predates the other.
Holling type II response	Model where the proportion of prey consumed declines monotonically with prey density.
Lokta-Volterra model	Model of competition between population involved.
Model	Mathematical representation of the reality by considering a set of assumptions.
Non-enrolled entities	Group of learners who did not enroll in the school for learning.
Parameter	Characteristic of a population.
Predator	Species that feed on the other (prey).
Prey	Species which being eaten/fed on by the predator.
Lyapunov function	Is vector valued function used to assess the global stability of the steady state.
Variable	A character which depends on the other at any given time.

ABSTRACT

The increase in Kenyan population attracted the establishment of more schools, both public schools and private schools. This was due to the need to cater for the increasing number of students being enrolled in schools. Moreover, the dynamics of students' population both in public schools and private schools have created the changes in the schools' population. This occurs through transfer from one category of school to the other, through completion of the learning period and through drop out due to unknown reasons. This subjected both the public schools and private schools to compete in order to maintain a good number of students under their custody. In this work, a modified Lotka-Volterra model of schools and non-enrolled entities population in the education system is studied. Private schools and non-enrolled entities play the role of a predator in public schools. Again, public schools and non-enrolled entities play the role of predators in private schools. This study uses integrated Holling type II functional response to analyze the model. Establishment of equilibrium points and their stability are determined using the Routh-Hurwitz criterion and eigenvalue method. Global stability has been done for the positive equilibrium point. Hopf bifurcation is also done around the positive equilibrium point. Data obtained from the Ministry of Education and the sources cited were used to estimate the model parameters. Finally, graphical illustration of various parameter is derived to show their effect on schools when they are varied. The study revealed that the increase in transfer rate from private to non-enrolled, transfer rate from public to non-enrolled and the non-enrolled entity predation on public schools greatly affects the schools' population as they are the ones leading to predation in school. Therefore, proper strategies should be developed to focus on reducing the parameters that affects the schools' population adversely to avoid leading schools' population to extinct.

CHAPTER ONE

INTRODUCTION

1.1 Background information

Both public and private schools exist to provide a learning environment to learners in the respective schools. According to (Alam & Tiwari, 2021), the past two decades indicates a rapid increase of 9.3 percent (196 million learners) in the share of private school enrolment between the year 2000 and 2019. However, there was drop outs from schools and transfer from one type of school to the other due reasons such as lack of fees, search of quality education, need to change environment of learning etc. When the COVID-19 emerged in the countries, both public and private schools faced closure. It was revealed that the pandemic hit private schools so hard compared to public school. When schools reopened, children from low income households faced difficulty in continuity of learning.

Mwakisisile & Mushi (2019), population is considered important element in regard to economic development. The learners in primary schools go through the process of teaching and learning for a period of eight years. Then they sit for national examination called the Kenya Certificate for Primary Education for which they proceed to secondary school for a period of four years. The examination being same in both private and public school. According to (Nishimura & Yamano ,2008), due to overcrowded classroom in public schools caused by the high enrolment, quality of education decreased resulting to transfer from public to private schools.

This study addressed public schools and private schools as the main categories of interest. The first category is the public schools which are owned and controlled by the government in their day to day operations. Teachers in such schools are government employees. The second category is private schools. Private schools had been defined by UNESCO as those only controlled by government but not operated nor managed by the government be it for profit motive or not. Thus falling under a private body for example community, foundation, faith-based organization, NGO, private proprietor or just a private enterprise (Alam & Tiwari, 2021). According to (Panja & Kumar, 2015), studying L-V systems has universal existence due to its importance and thus as a subject its dominant in the sector of mathematical ecology. Thus for conformity in learning programs, the two categories of schools depend on each other. Traditionally, there were no private schools and hence public schools would gain students through the area population. Private schools tend to concentrate in an area where there is poor performance of public schools. However, according to Talance (2020), the development of private schools increased the competition in the schooling system. This led to theoretical efficiency gain in terms of both quality and costs as private and public institutions compete to attract students.

Due to the prevailing situations where the country is geared towards achieving the Sustainable Development Goals (SDG) and the country's goals dubbed 'Big four agenda' which in addition is aimed at realizing the vision 2030. The country's activities are thus focused towards realization of SDG and big four agenda through better education system. Going down to the main four agenda of the Kenyan government namely food security, housing accessible to citizens, universal health cover and manufacturing and employment through securing job could only be realized when the education system is improved. Thus good education system promotes the achievement of SDG. Thus parents who can afford to pay private school fees would transfer their children to a school where they believe provides quality education.

In Kenya, both private schools and public schools emerged and coexisted with some degree of competition in most parts of the country and thus the competition that exist in the two categories of school is of good study for the future prospect of the schools.

Literature review revealed that much has been done on schools' population by social statistical studies. But till now very little work has been done on population dynamics of private schools, public schools and non-enrolled entities population together. Therefore, the main objective of this study is to formulate a mathematical model using Lotka-Volterra equations that can help to analyze the students' populations in the categories of schools and non-enrolled entities and the dropouts.

The analysis of prior studies discussed the factors that could lead to transfer and not considering the future of them. Thus the secondary objectives of this study was to analyze the model and finally simulate the model.

The Lotka-Volterra system was proposed by the American ecologist Lotka in 1921 and by the Italian mathematician Volterra in 1923 (Mao *et al.*, 2020). Lotka-Volterra is basically based on two species equations but this study comprised of three equations hence a more advanced model of study. Hung *et al*. (2017), the two species equations of Lokta-Volterra can be two competing technologies, two competing products, two competing channels, or two competing species in an ecosystem. The L-V model is used where two or multiple species compete for the survival purpose. Lotka-Volterra is used to forecast the population of each species and reveal the interaction effects among them. This study determined the dynamical behavior of the schools' by performing the stability and bifurcation analysis of the Lotka-Volterra systems.

1.2 Statement of the problem

According to Ministry of Education (2019), in Kenya public secondary schools possess a large number of students compared to private secondary schools. The main factor that had contributed high population in public secondary schools was the low cost of education and their easy accessibility both in urban and rural areas. However, there has been movement of students from one category of school to the other by transfer and vice versa. Transfer has been occasioned by several factors like performance, peer influence, family economic situation changes. (Talance, 2020), the "differentiated demand" model, private and public schools are imperfect substitutes. Parents preferred private schools because they want their children to acquire specific characteristics such as a higher quality education, religious courses or a specific language of instruction. In regard to achieving one of the big four agenda, that is, manufacturing and job creation for example parents needed their children to be creative after school completion. Despite the fact that the above aforementioned factors are considered as the causes of transfer between public and private school, very little attention has been given to mathematical model to account how these factors influenced the schools' populations. It is from this background that this study opted to formulate a mathematical model for school students' population dynamics using Lotka-Volterra equations incorporating the Holling type II response.

1.3 Justification of the study

A range of social statistical techniques have been used to explain schools' population dynamics (Duncan & Sandy, 2007; Baum & Riley, 2019). However, it was important that approaches to modelling school population dynamics be considered based on past enrolment data incorporating the effects of competition and predation.

Mwakisisile & Mushi (2019), mathematical models are very useful for projecting the total population. Thus mathematical models are of fundamental use when it comes to explore schools' population dynamics and project future enrolment in schools. This being most important tasks for educational planners. Predicting the future population is a core factor for development planning and decision making, Mwakisisile & Mushi (2019). Thus the government can be able to make proper arrangement for the predicted situations and be able to achieve quality education as one the SDG of the country. The findings of the study are used to explore the factors attributed to school transfer and further discuss the implications for future modelling by education strategist.

1.4 Significance of the study

Students transfer from one school to the other has an effect on the students' performance. It can lead to improvement or a decrease in performance. The transfer also has effect to government in management of public schools in terms of disbursement of money to cater the student education because the government disburse money once in a term or a year. Therefore, transfer from private to public resulted to economic burden which are unbearable in the school. This study investigated students' population in public, private school and those students who are not in school but should be in school. The study used mathematical model to study the dynamics of the students. By determining of effects of parameters such as influence, unknowns' factors, carrying capacity of the two schools it is helpful on how to control them. Thus education stakeholders and government can capture and reduce dropout rate, influence rate by taking well established measures. It also help in the future prediction of the student population in schools (Mwakisisile & Mushi, 2019).

1.5 Research questions

1. How does schools' population change with the effect of Holling type II response?
2. Are the competition models on schools' population and the non-enrolled entity well-posed?
3. What is the trend of schools' population dynamics?

1.6 Objectives of the study

1.6.1 General objective

To model, analyze and simulate Lotka-Volterra model of competition among various categories of schools in Kenya incorporating Holling type II response.

1.6.2 Specific objectives

The specific objectives are to:

1. Formulate a hybrid logistic and exponential mathematical model using Lotka-Volterra equations incorporating predation effect of Holling type II response.
2. Analyze the model using theories of ordinary differential equations.
3. Simulate the model using Matlab inbuilt ordinary differential equation solvers.

1.7 Scope of the study

This study was done on Kenya basic education schools' population. Thus public primary schools and private primary schools in Kenya were considered in model development. The first two types of schools formed two variables x and y. The model also considered pupils who did not enrolled in school, forming the third variable z as the non-enrolled entity The research was used to evaluate the transfer rate in schools depending on economy, influence and unknown factors.

CHAPTER TWO

REVIEW OF LITERATURE

2.1 Introduction

This chapter comprised of the literature review on the previous research done schools. The used Lotka-Volterra to study dynamics of schools' population. Thus literature review has been done on the previous use and applications of Lotka-Volterra model.

Achieving Education for All (EFA) was put forward by many developing countries and some countries had gone further to adopt the Universal Primary Education (UPE). Since then, private education has flourished in many parts of Africa. Many researchers have tried to investigate the reason behind private schooling type. Some collected data from the school participants and parents and analyzed the data. From the findings, private schooling emergence was easy to point out though very difficult to predict the future prospect of the public and private schools.

2.2 Empirical review

Duncan & Sandy (2007) looked at the factors explaining the performance gap between public and private school students. Survey data was used and the decomposition of the findings indicated that the gap was due to differences in individual, household type and school features. Revealing that privately educated individuals had more of their attributes associated with high test performance.

Nishimura & Yamano (2008) studied emerging private education in Africa. A community survey was done. The results indicated that enrolment increased in private schools immediately after the introduction of FPE because quality deceased in public schools and thus parents who were able to pay relatively high school fees opted for private schools.

Talance (2020) investigated why some parents choose private schools when there are free public schools. The results were explained by the 'excess demand model' which argued that due to budgetary and space constraints, parents opt for private school because public school cannot meet expanded demand for education. Furthermore, the 'differentiated demand model' pointed out that some parents choose private school because they want their children to acquire some special characteristics such as higher quality education, religious course etc.

Reimann *et al.* (2020) investigated the competition between public and private primary schools in Berlin focusing on the perception of competition and the process of student selection. The findings revealed that schools with good reputation are more likely to benefit from competition because good reputation increases the demand for spots at that school. Effects of change of factors increasing school demand were not considered.

2.2.1 Mathematical model formulation using Lotka-Volterra with predation.

Sooknanan *et al.,* (2016) studied a modified L-V model which was used to reveal interaction between police officers and gangs members. The model of eco-epidemiological was used for formation of the interaction. It considered the consequences brought by corruption as a results of the predation between gang members and police populations. The developed Modified Lotka-Volterra model which had the predator represented by the police and the prey represented by the gang members was analyzed. The gang members were mostly associated with predation effect while police continue hunting but at a reduced rate. Assumptions were made and then a system of equations using ODEs formed. The systems were analyzed and stability done by linearizing the system by varying the change in equilibrium points. The stability analysis and bifurcation was done numerically using MATLAB. Media reports were used due to difficulty in accessing official data from police. The findings of the study revealed that parameters which were identified can varied which would bring a great change leading to reduction in the gang members.

Hung *et al.*, (2017) looked at modified application of L-V model to forecast sales of two competing retail formats. The retail formats were convenience oriented format and budget oriented format. The proposed approach used sales data as compound data when decomposed into three; aggregate, competition and seasonal component. L-V model was utilized in forecasting the competition component as opposed to the traditional approach which used compound data as the input into the Lotka-Volterra model. In comparison of forecasting errors empirical study was also used. The results of the said comparison from the empirical study revealed that the proposed approach was much better than the traditional approach in terms of forecasting errors. Predator- prey relationship was formed only that the role of predator was reversed. The proposed approach further proved that convenience-oriented format was the predator as its benefits were more preferable with the growth of GDP with time.

Tie Wei, Zhiwei Zhu & To (2018) "examined the Lotka-Volterra model to assess the dynamic process and evolution of innovation resource competition." The non-zero solutions at maximum resource was considered i.e. at equilibrium and looked at the graph of the solutions. In the graph, further analyzed the crowding out effect, stable equilibrium and non-stable equilibrium. When $\frac{dx_1}{dt} > 0$ and $\frac{dx_2}{dt} < 0$, meaning x_1 increases as x_2 decreases with time. x_1 has ability to compete for resource than x_2 and thus has resource crowding effect than x_2 and will completely crowd x_2 out of competition. It looked at the possibility of achieving stable and unstable equilibrium. When the model's parameters were changed, it was discovered that the initial competitive ability was stronger when the natural growth parameter was larger. The end consequence of evolution overwhelmed by higher aggressive enterprise, was not altered.

Li *et al.*, (2018) studied the control optimization in L-V model. The model considered the control of linear feedback. It also considered Integrated pests control system which utilizes reliable methods and techniques comprehensively to lower side effects to economy as a result of pests. It used a logistic growth to form two impulsive differential equations of pests and pesticides. Differential equation theory was used to prove the necessary conditions of well-posed and stable system. Optimization was done to help in the reduction of total cost to be incurred in pest control.

Tahara *et al.,* (2018) studied asymptotic stability of modified L-V model with a small immigration. Ordinary differentials of the nonlinear equations were used to form the equation of prey and predator. The asymptotic stability was analyzed in the presence of small addition of immigration in both the population of prey and predator. Several cases of migration and immigration factor were considered for the long term effect of population dynamics. The findings revealed that immigration into the prey and predator caused a stability effect while migration caused destabilization.

(Khan et al., 2018) studied employing the Lotka-Volterra model, researchers examined the dynamic effects of low-cost carriers on full-service carriers and the South Korean tourism industry. Jeju Island the only destination without substitute mode of fast transport. Two systems of non-linear ODE were formed. The least square method was used to calculate the interacting parameters of the non-linear. The air demand data obtained from the air travel portal was used. The modified Lotka-Volterra model forecasted results were compared with SARIMA, which is a competent one, and it was established that the results were better than SARIMA model and it also performed better in terms of applicability than the performing SARIMA.

(Zhou *et al.*, 2019) studied L-V equations and came across with a system with functional response.

$$\frac{dx}{dt} = v(x) - yu(x) \tag{2.2.1.1}$$

$$\frac{dy}{dt} = -\mu y + \eta yu(x) \tag{2.2.1.2}$$

The variable $x(t)$ denoted the density of prey while $y(t)$ denoted the density of the predator at any given time t. Given that μ, η were positive constants which represented the death rate of predator and the consumption rate of prey respectively. Hence in the absence of predators $v(x)$ is the growth rate of the prey population while $u(x)$ is the functional response. The functional response represented how many predators the prey could catch in a unit of time.

Mao *et al., (2020)* studied L-V model of two competitive species populations in an ecosystem. Populations can be represented by $X_1(t)$ and $X_2(t)$ and therefore the mutual relationship was represented by L-V model as follows,

$$\frac{dx_1}{dt} = a_1 x_1 - b_1 x_1^{\,2} - c_1 x_1 x_2 \tag{2.2.1.3}$$

$$\frac{dx_2}{dt} = a_2 x_2 - b_2 x_2^{\,2} - c_2 x_1 x_2 \tag{2.2.1.4}$$

where a_1 and a_2 is the growth rate for species x_1 and x_2 without staying together, b_1 and b_2 are the inhibitor in their environment. Parameters c_1 and c_2 are the association parameters. Competition involved depended on the sign of the parameters c_1 and c_2

Mao *et al*., (2020) used Grey L-V equations to model the Competition and Cooperation. This was between third-party online payment systems and online Banking in China. Accuracy was improved by introducing a background coefficient. The parameters of the model were estimated by least square method and the Grey-Lotka-Volterra model for fitting the relationship between third-party payment and bank competition and cooperation is obtained. Online banking is the predator while third party payment is the prey. The results showed that third party continues to grow steadily due to growth of transaction scale in mobile payment industry which had maintained a large scale base. It also revealed that there is a mutual inhibition between banks and third party payment and banks inhibition is greater than that of online third party online payment.

S. Y. Wang *et al*. (2021) studied competition on industries population with the help of a three dimensional Lotka-Volterra model. System of equations were formed using ODEs. Jacobian matrix and the trace of Jacobian matrix was used to analyze the local stability. Three automobile industries were chosen as the research objects for analysis. It discussed the effect of change of parameters, and the relevant parameters were revealed by regression analysis. The findings of the study revealed that steady state condition is possible for both coexistences. The analysis for the coexistence was done and it revealed that cooperative action is better than competitive strategy and that the method is more practical. The results revealed that the approach is feasible and effective in examining the rivalry, progress, and balanced growth of the enterprise population.

2.2.2 Carrying out the model analysis

Panja & Kumar (2015) studied the stability analysis of coexistence of three species. Three non-linear equations of competition were formed with the logistic growth and Holling type II response. Zooplankton, phytoplankton and fish were used to form predator-prey model. The three species interchanged in the role prey and predator like fish is the predator to Zooplankton and zooplankton is the predator to phytoplankton. Assumptions were made and Routh-Hurwitz method was used to analyze the local stability of equilibrium points. Lyapunov method was used to analyze the global stability. Hopf bifurcation and sensitivity analysis was done to show variation of parameters.

H. T. Wang & Wang (2016) studied diffusion process and discussed the dynamic of Android and iOS which were in sales volume competition. Two systems of ODEs were formed by taking into account three variables: the size of the market niche, the size of the attractiveness of the product, and the interactions that occurred during competition. iOS is the predator in the connection between Android and iOS, and Android is the prey. While iOS had a negative impact on Android's sales volume, Android's sales growth was positively impacted by iOS. Jacobian matrix and Lyapunov function were used to examine the stability of the equilibrium points; the results were negative, showing that both products coexist in the market. It was determined that the actual sales volume and the predicted sales volume are pretty close, which suggests that based on model fitness, the L-V model demonstrated a significant explanatory power.

Tsai (2017) studied utilizing the L-V model of the logistic equation in prediction of the competitive relationships between Taiwan and China in the industrial manufacturing of motherboards. All of the key variables that had an impact on the shipment of Taiwanese and Chinese motherboards were divided into two systems of equations. In order to optimize the parameters, the proposed L-V model generated two nonlinear differential equations that were numerically solved using an algorithm of the genetic (G.A) methodology and simultaneously. Jacobian matrix and Lyapunov function was used to analyze the equilibrium points and was established to be negative from the analytical values satisfying the (Hritonenko & Yatsenko 1999) stability condition. Quarterly global data obtained from the MIC database was used. The results showed that the procedure generates parameters that are stable and trustworthy and only slightly vary and their exist a predator-prey relationship between Taiwan and China; Taiwan is the predator while China is the prey. Analysis revealed that Lotka-Volterra is more accurate in forecasting shipment performance than the Bass model's MAPE. The method provided a long term relation of the two.

Pei *et al*. (2017) used Grey Tripartite Lotka-Volterra model in the study of estimation of competitive relationships between Amazon, Alibaba and Suning. Continuous Lotka-Volterra model is transformed to discrete model. Eight equilibrium points were obtained which the stability was analyzed using the Jacobian matrix. The findings revealed that Alibaba gains market share from Suning and Amazon, so increasing sales of Alibaba weaken the performances of the other two e-commerce companies because of negative parameter in those companies. The sample data was used for the future prediction and when compared to MAPE, Grey Tripartite Lotka-Volterra showed less errors. The predicted future results were that Alibaba threatens the development of Suning and Amazon, while the operating statuses of Suning and Amazon cannot influence the development of Alibaba.

Avila *et al.,* (2018) examined the connection between competition and diffusion. To make the competitive interaction between mobile and fixed line services clearer, Lotka-Volterra model was used. Findings revealed that fixed line service mobile telephone initially acted as competitors with equal opportunities. However, as mobile service became available almost nationwide, the relationship of the two changed to amensalism

Gavina *et al.* (2018) studied the multispecies coexistence in Lotka-Volterra competitive systems with crowding effects. The authors modified the classical Lotka-Volterra by including the nonlinear crowding effects. Assumption were made of high mortality at high density. The results were compared with aggregation model of coexistence. The findings revealed that the aggregation model becomes difficult to maintain the coexistence of all or many species when the number of species was increased. In contrast, the coexistence of all or many species can be easily achieved in the current crowding model even when the number of species was increased.

Agmour *et al.* (2018) studied the stability analysis of a competing fish population model in the presence of the predator. Three logistic non-linear equations were formed. Boundedness and positivity of the solution were looked at. Routh-Hurwitz method was used to analyze the stability of different equilibrium points. Numerical simulation was done to verify the stability and instability of eight equilibrium points.

Zhang *et al*. (2019) studied system evolution prediction and manipulation. To depict the link between a system and its parts, the Lotka-Volterra model was utilized. The authors formed systems of ODEs. The results were solved numerically. The parameters in the model were associated with their casual factors e.g. R and D Investment, government change of policy which offers guidelines to designers to improve system performance. Data fitting was done to determine the optimal values of the parameters.

Kawira *et al*. (2020) studied corruption between staff and students in higher institutions of learning using Lotka-Volterra model. Corrupt students and staff acted as the predator while non- corrupt as the prey. Corruption was looked at using the epidemic diffusion model of Kermack and Mc-Kendrick (1927) and determined the dynamics of corruption and its effects on the composition of the population at a given time. The population was divided into compartmental models. Differential equations were formed and analysis done for boundedness, feasibility and the steady states. Jacobian matrix and Lyapunov method was used to determine the local and global stability respectively. From the numerical bifurcation analysis, it was established that the corrupt student would decrease with time while corrupt staff would increase with time because staff stay longer in an institution than students. Thus, any corruption mitigation strategy should be focused more on staffs than students.

2.3 Knowledge gap

Studies on schools have been done to show the causes of students transfer from a public school to a private school and vice versa. Talance (2020) stated that parents consider school quality, as explained by excess demand model and differentiated demand model. This resulted to the emergence of private education as due to the deteriorated quality of public schools. There are other factors other than performance that causes changing of school by students.

Very limited research has been done to show competitive relationship between two categories of schools. Therefore, there is an evidenced need to do research study on the two categories of schools to show student population dynamics in public and private school. The findings are of fundamental use to education stakeholders or experts, the government and parents to help in preventing the dropout rate.

CHAPTER THREE

MATERIALS AND METHODS

3.1 Model formulation

This chapter discusses how the model was developed and assumptions made in coming up with the model. The chapter also introduces the basic concepts how the model will be analyzed. The study presents relationship between three different variables inform of a system of differential equations. Modified Lotka-Volterra model was formulated that is used in modelling the competition between three species incorporating the Holling type II response. Private schools' population grew by enrolment and transfer from public schools and again the predator (non-enrolled) gain population by decreased enrolment in schools and drop outs. Private schools and non-enrolled entities assumed the role of a predator while public schools were the prey. Model formulated described the dynamics of public schools, private schools and the non-enrolled entities population. The model adopted the Lotka-Volterra equation incorporating the Holling type II response. This is because of low prey population density, thus illustrating that the level of predation is high where the level of consumption reaches a saturation point, thus the model with the Holling type II functional response is adopted according to (Pusawidjayanti *et al.*, 2021). The total recruitment into the said categories is denoted by π which was subdivided into the following categories $\gamma_1\pi$ as the recruitment in public schools, $\gamma_2\pi$ as the recruitment in the private schools and $(1-\gamma_1-\gamma_2)\pi$ for the non-enrolled entities. Since public and private schools considered their carrying capacity at the time of enrolment, their model was logistic growth as opposed to non-enrolled entities which assumed an exponential growth. The population dynamics in the categories was due to predation effect (Holling type II response), exit rate and the competition as represented by η, μ and θ respectively. The exit rate was a fraction of the total population and the study period. For the non-enrolled entities, the study considered their exit rate as the period which a person cannot come back to school to study. Anybody who had not joined the two category of schools was considered to be in the non-enrolled category even if they joined tertiary institutions. Since the model dealt with basic education (primary and secondary education), the model considered the number of years taken to complete primary and secondary education.

3.2 Assumptions of the model

In addition to fundamental assumptions of Lotka-Volterra, the model also assumed the following;

i. The study does not include special schools and group of schools in the country.
ii. Dropout rate due to other factors other the ones discussed in the model was insignificant.
iii. Private schools are purely funded by only the parents of students in that school.
iv. The rate of student retention in a class was insignificant.
v. Public and private schools cannot predate the non-enrolled entities.

3.3 Model variables and parameters

This section outlines the model parameters and variables used in the model formulation of the study.

Table 1: Model variables

Variable	**Description**
$x(t)$	The population of students in public schools at any given time t.
$y(t)$	The population of students in private schools at any given time t.
$z(t)$	The population of those who dropped out of school and those did not get enrolled at any given time t.

Table 2: Model parameters

Parameter	Description
π	Enrolment rate in schools.
$\pi\gamma_1$	Enrolment rate in public schools.
$\pi\gamma_2$	Enrolment rate in private schools.
B_1	Constant of the saturation (0.5).
B_2	Constant of the saturation (0.5).
B_4	Constant of the saturation (0.5).
Θ_1	Transfer from public schools to private due to unknown factors.
μ	Exit rate due to completion.
K_1	Public schools carrying capacity.
K_2	Private schools carrying capacity.
Θ_2	Transfer from private schools to non-enrolled entity due to unknown factors.
Θ_3	Transfer from public schools to non-enrolled entity due to unknown factors
B_3	Half saturation constant.
η_1	Private schools predate public schools
η_2	Non-enrolled entities predate private schools.
η_3	Non-enrolled entities predate public schools.
η_4	Public schools predate private schools.

3.4 Model flow chart

Model flow chart below represents the interactions of various variables and parameters. It captures the movement of individuals in the system in and out as shown in Figure 1.

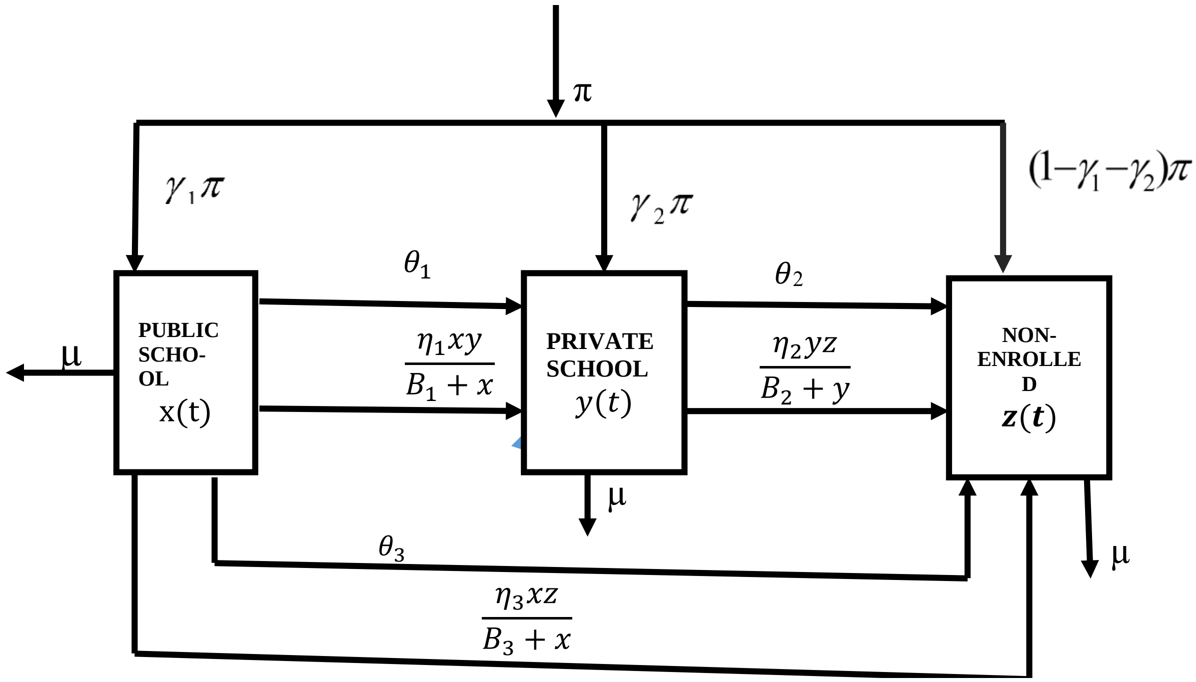

Figure 1: Model flowchart

3.5 Model Equations

Based on the above model flowchart, model description and model assumptions, three ordinary differential equations were deduced. The variable $x(t)$ represents the public schools' population at any given time t, $y(t)$ represents private schools' population at any given time t and $z(t)$ represents the populations of those students who are not in schooling at the moment but they are of school going age. The derivatives $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ mean the instantaneous rate of change of populations of public schools, private schools and the non-enrolled entity category represented by X, Y and Z respectively at time t.

$$\frac{dx}{dt} = \gamma_1 \pi x(1 - \frac{x}{K_1}) - (\mu + \theta_1 + \theta_3)x - \frac{\eta_1 xy}{B+y} - \frac{\eta_3 xz}{B+x} + \frac{\eta_4 xy}{B+y} \tag{3.5.1}$$

$$\frac{dy}{dt} = \gamma_2 \pi y(1 - \frac{y}{K_2}) - (\mu + \theta_2)y + \frac{\eta_1 xy}{B+y} - \frac{\eta_2 yz}{B+y} + \theta_1 x - \frac{\eta_4 xy}{B+y} \tag{3.5.2}$$

$$\frac{dz}{dt} = (1-\gamma_1-\gamma_2)\pi z + \theta_3 x + \theta_2 y + \frac{\eta_3 xz}{B+x} + \frac{\eta_2 yz}{B+y} - \mu z \qquad (3.5.3)$$

3.6 Model analysis

This was done by showing the positivity of the model, boundedness of the solution, determining the equilibrium points and the stability of the equilibrium points.

3.6.1 Investigation of the feasibility of the model and boundedness of the solution

Systems of ODEs with initial conditions of the variables x, y and z which are ≥ 0 at time t, were considered. The analysis helped to ensure that the model is meaningful, well-posed and that it is realistic in representing the human population. Panday *et al.,* (2018), since the solution is bounded, none of the interacting species experienced long-term exponential or sudden growth. A proof was done to show that all solutions of the system of equations with positive initial data remained positive for all time and are bounded in the feasible region of the model (Real domain).

3.6.2 Determination of the equilibrium points

The equilibrium points of the system were determined by equating the system of equations (3.5.1), (3.5.2), (3.5.3) to zero.

3.6.3 Investigation of the local stability of the equilibrium points

The local stability of the steady states is determined by the nature of the eigenvalues of the Jacobian matrix. (Kawira *et al.,* 2020), reduce non-linear system to linear thereby varying the equilibrium points slightly. Jacobian matrix of the systems of equations of (3.5.1), (3.5.2), (3.5.3) were checked then Routh-Hurwitz criteria was used to determine local stability.

3.6.4 Global stability investigation in the system

Global stability analysis was done by constructing a Lyapunov function. To determine whether an equilibrium point of a nonlinear system is globally stable, examination of the stability of the linear approximation to the nonlinear dynamics about the equilibrium point is done. This approach is known as Lyapunov method (Keya & Islam, 2020).

3.6.5 Performing of the bifurcation analysis

Kawira *et al*., (2020) bifurcation analysis is done to consider the behaviors of a given parameter in the long run. Panja & Kumar (2015) types of dynamical behaviors may occur and the critical parameter values at which such transitions happen are called bifurcation points. Bifurcation helped to study the qualitative behavior of the dynamical system as a result of a parameter value change (bifurcation parameter).

3.7 Numerical simulation

This section presents a series of numerical results obtained from simulation using MATLAB. Wolfram Mathematica software was used to analyze the systems of equations. According (Panja & Kumar, 2015) the data for the few parameter which were unavailable, the research considered using hypothetical values for them.

CHAPTER FOUR
RESULTS AND INTERPRETATION

4.1 Overview

The results and the interpretation of the study are presented in this chapter. The numerical simulation of the model performed using data obtained from the literature review and estimation has been presented. The parameter which brought significant changes or are influential to the model has been demonstrated by varying the value of the parameter.

4.2 Model analysis

The analysis of the model was done to check if the model lies in their feasible region, determining the equilibrium points and analyzing their stability. Furthermore, bifurcation was done around the positive equilibrium.

4.2.1 Feasible region

Since the model considers human population, it is important to show that all its state variables are positive for all future time, t.

Consider equation (3.5.1),

$$\frac{dx}{dt}=\gamma_1\pi x(1-\frac{x}{k_1})-(\mu+\theta_1+\theta_3)x-\frac{\eta_1 xy}{B+y}-\frac{\eta_3 xz}{B+x}+\frac{\eta_4 xy}{B+y}$$

by inspection method the term $\frac{\eta_4 xy}{B+y}\geq 0$ and $\gamma_1\pi x(1-\frac{x}{k_1})\geq 0$. Thus

$\frac{dx}{dt}\geq-(\mu+\theta_1+\theta_3)x-\frac{\eta_1 xy}{B+x}-\frac{\eta_3 xz}{B+x}$. It follows,

$\frac{dx}{x}\geq-((\mu+\theta_1+\theta_3)-\frac{\eta_1 y}{B+x}-\frac{\eta_3 z}{B+x})dt$. On integrating,

$\ln x\geq-((\mu+\theta_1+\theta_3)-\frac{\eta_1 y(s)}{B+x(s)}-\frac{\eta_3 z(s)}{B+x(s)}ds+c)$.

$$x(t)\geq x(0)e^{\int_0^t-((\mu+\theta_1+\theta_3)+\frac{\eta_1 y(s)}{B+x(s)}+\frac{\eta_3 z(s)}{B+x(s)})ds}.$$

Exponential of a negative is always positive. Thus $x(t)$ stays positive for all time t.

Consider equation (3.5.2),

$$\frac{dy}{dt}=\gamma_2\pi y(1-\frac{y}{k_2})-(\mu+\theta_2)+\frac{\eta_1 xy}{B+x}-\frac{\eta_2 yz}{B+y}+\theta_1 x-\frac{\eta_4 xy}{B+y},$$

by inspection method, the term $\gamma_2\pi(1-\frac{y}{k_2}),\frac{\eta_1 xy}{B+x}$ and $\theta_1 x$ is ≥ 0.

Implying that $\frac{dy}{dt}\geq -((\mu+\theta_2)y+\frac{\eta_2 yz}{B+y}+\frac{\eta_4 yx}{B+y})$,

Integrating this as follows

$$\frac{dy}{y}\geq -((\mu+\theta_2)+\frac{\eta_2 z}{B+y}+\frac{\eta_4 x}{B+y})dt$$

$$\ln y\geq -((\mu+\theta_2)+\frac{\eta_2 z(s)}{B+y(s)}+\frac{\eta_4 x(s)}{B+y(s)})ds+c$$

Thus, $y(t)\geq y(0)e^{\int_0^t -((\mu+\theta_2)+\frac{\eta_2 z(s)}{B+y(s)}+\frac{\eta_4 x(s)}{B+y(s)})ds}\geq 0$. i.e. the exponential of a negative is positive. Thus $y(t)$ stays positive for all time t.

For equation (3.5.3), $\frac{dz}{dt}=(1-\gamma_1-\gamma_2)\pi z+\theta_3 x+\theta_2 y+\frac{\eta_3 xz}{B+x}+\frac{\eta_2 yz}{B+y}-\mu z$

Similarly, by inspection method the term

$$(1-\gamma_1-\gamma_2)\pi z+\theta_3 x+\theta_2 y+\frac{\eta_3 xz}{B+x}+\frac{\eta_2 yz}{B+y}\geq 0.$$

Thus $\frac{dz}{dt}\geq -\mu z$.

Implying that $\frac{dz}{z}\geq -\mu dt$. On integrating, results to

$z(t)\geq z(0)e^{-\mu t}\geq 0$.

Thus $z(t)$ stays positive for all time t.

4.3 Boundedness of the solution

The boundedness of a function is about finite limits. A function has upper limit and lower limit of which it doesn't go beyond incase of upper limit or below in case of lower limit. Since the population is not constant, a positively invariant region was obtained, in which the model solution is bounded.

For the system to be mathematically meaningful, it is necessary to show that its state variables are positive and bounded for all time t (Panja & Kumar, 2015). That is, the solution of the system with a positive initial values remained positive for all time $t \geq 0$.

The total population is given by,

$$N = x + y + z$$

$$\frac{dN}{dt} = ((\gamma_1 \pi x(1-\frac{x}{k_1}) - (\mu+\theta_1+\theta_3)x - \frac{\eta_1 xy}{B+x} - \frac{\eta_3 xz}{B+x} + \frac{\eta_4 xy}{B+y} + (\gamma_2 \pi y(1-\frac{y}{k_2}) - (\mu+\theta_2)y$$

$$+\frac{\eta_1 xy}{B+x} - \frac{\eta_2 yz}{B+y} + \theta_1 x - \frac{\eta_4 xy}{B+y}) + ((1-\gamma_1-\gamma_2)\pi z + \theta_3 x + \theta_2 y + \frac{\eta_3 x}{B+x} + \frac{\eta_2 yz}{B+y} - \mu z)).$$

On simplifying the above,

$$\frac{dN}{dt} = \gamma N(1-\frac{N}{k_4}) - \mu N .$$

$$\frac{\log N - \log[N\gamma + (-\gamma+\mu)k_4]}{\gamma-\mu}$$

$$N(t) = \frac{Ce^{t\gamma}(\gamma-\mu)k_4}{-e^{t\mu} + e^{t\gamma}C\gamma},$$

$$t = 0, \quad C = \frac{N(0)}{-\gamma k_4 + \mu k_4 + \gamma N_0}.$$

Thus , $N(t) = \dfrac{e^{t\gamma}(\gamma-\mu)k_4 N_0}{e^{t\gamma}\gamma N_0 - e^{t\mu}(1-\gamma+\mu)k_4 + \gamma N_0}$.

4.4 Steady states

In the analysis of predator- prey model or differential equations in general, the state variables that do not change with time are more necessarily to be considered. That is there is no change of situation in the system. These solutions are referred to as the equilibrium points. Thus, the steady states are obtained by equating system of equations in (3.5.1), (3.5.2), (3.5.3) to zero.

Therefore, for the sake of studying the stability of the proposed model, the equilibrium points of the system needed to be determined. The study therefore considered the steady states of the variational matrix below under different cases as discussed below.

$$\begin{pmatrix} -\mu+(\pi-\frac{2\pi x}{k_1})\gamma_1-\frac{yB\eta_1}{(x+B)^2}-\frac{zB\eta_3}{(x+B)^2}+\frac{y\eta_4}{y+B}-\theta_1-\theta_3 & -\frac{x\eta_1}{x+B}-\frac{xy\eta_4}{(y+B)^2}+\frac{x\eta_4}{y+B} & -\frac{x\eta_3}{x+B} \\ \frac{yB\eta_1}{(x+B)^2}-\frac{y\eta_4}{y+B}+\theta_1 & -\mu+(\pi-\frac{2\pi y}{k_2})\gamma_2+\frac{x\eta_1}{x+B}-\frac{zB\eta_2}{(y+B)^2}-\frac{xB\eta_4}{(y+B)^2}-\theta_2 & -\frac{y\eta_2}{y+B} \\ -\frac{xz\eta_3}{(x+B)^2}+\frac{z\eta_3}{x+B}+\theta_3 & -\frac{yz\eta_2}{(y+B)^2}+\frac{z\eta_2}{y+B}+\theta_2 & -\mu+\pi\alpha+\frac{y\eta_2}{y+B}+\frac{x\eta_3}{x+B} \end{pmatrix}$$

Case 1: **No schools exist**

In the absence of schools, then it implies that $x=0$, $y=0$, $z=0$. The equilibrium point is then denoted as E^0. Thus if the population does not exist then the equilibrium point is obtained by setting the variables x, y and z to zero. Consequently, the right hand side of equations (3.5.1), (3.5.2) and (3.5.3) are set to zero to obtain a trivial solution which is $E^0=(0,0,0)$.

Case 2: Only public schools exist

The equilibrium point where only public schools exist is denoted as E^1. In such a case the variables $x\neq 0$, $y=0$ and $z=0$. As a results equation (3.5.1), (3.5.2) and (3.5.3) are reduced to $\gamma_1\pi x(1-\frac{x}{k_1})-(\mu+\theta_1+\theta_3)x=0$.

Thus we have E^1 being given as $E^1 = (x^1, x^0)$ where $x^0 = 0$ and $x^1 = \frac{(k_1(\mu - \pi\gamma_1 + \theta_1 + \theta_3)}{\pi\gamma_1}$.

Case 3: Only private schools exist

The equilibrium point where only private schools exist is denoted as E^2. In such a case the variables $x = 0$, $y \neq 0$ and $z = 0$. As a results equation (3.5.1), (3.5.2) and (3.5.3) are reduced to $\gamma_2 \pi y(1 - \frac{y}{k_2}) - (\mu + \theta_2) y = 0$. Thus the steady state E^2 is given as $E^2 = (y^2, y^0)$ where $y^0 = 0$ and $y^2 = \frac{k_2(-\mu + \pi\gamma_2 - \theta_2)}{\pi\gamma_2}$.

Case 4: No enrolment in schools

In the absence of total enrolment in schools, the variables x, y and z are set as $x = 0$, $y = 0$, $z \neq 0$. As a results equations (3.5.1), (3.5.2) and (3.5.3) reduces to $(1 - \gamma_1 - \gamma_2)\pi z - \mu z = 0$. The equilibrium point were then be denoted by E^3 where $E^3 = z^0$ and when $\alpha = \mu$, where $z^0 = 0$.

Case 5: Absence of private schools only

In a case where students either join public schools or not joining at all, the variables x, y and z are represented as $x \neq 0$, $y = 0$, $z \neq 0$. Therefore, upon substituting and solving for the equilibrium, three equilibrium points are obtained namely E_0^4, E_1^4 and E_2^4.

The three equilibrium points are obtained are:

i) $E_0^4 = (x_0^4, z_0^4)$ where $x_0^4 = 0$ and $z_0^4 = 0$,

ii) $E_1^4 = (x_1^4, z_1^4)$ where,

$$x_1^4 = \frac{m-n}{q}, \quad z_1^4 = \frac{(m-n)\left(\left(-\frac{m-n}{q} + k_1\right)\alpha_1 + k_1\beta_8\right)}{qk_1\alpha_5}.$$

iii) $E_2{}^4 = (x_2{}^4, z_2{}^4)$,where $x_2^4 = \dfrac{m+n}{q}$ and $z_2^4 = \dfrac{(m+n)\left(\left(-\dfrac{m+n}{q}+k_1\right)\alpha_1 + k_1\beta_8\right)}{qk_1\alpha_5}$

where

$$m = \alpha_1\alpha_5 + k_1(\alpha_1 + \beta_8)\varphi_3,$$

$$n = k_1\alpha_5\sqrt{\frac{4k_1\alpha_1\alpha_5(-\alpha_1+\beta_1)\varphi_3 + \left(\alpha_1\alpha_5 + k_1(\alpha_1+\beta_8)\varphi_3\right)^2}{k_1^2\alpha_5^2}}$$

$$q = 2\alpha_1\varphi_3 \ , \ \alpha_5 = \mu - \alpha_4 \ , \ \beta_5 = \frac{\mu - \alpha_4}{\theta_3} \ , \ \beta_6 = \alpha_1\beta_5 \ , \ \beta_7 = \beta_1\beta_5 \ ,$$

$\beta_8 = -\beta_1 + \theta_3$ and $\varphi_3 xz = \dfrac{\eta_3 xz}{B+x}$, $\alpha_4 = (1-\gamma_1-\gamma_2)\pi$,

$$\beta_1 = \mu + \theta_1 + \theta_3, \ \alpha_1 = \gamma_1\pi \ , \ \alpha_2 = \gamma_2\pi \ .$$

Case 6: Full enrolment

For the full enrolment in schools the variables $x \neq 0$, $y \neq 0$ and $z = 0$.

By setting the variables as shown, three equilibrium points are obtained;

$E_0^5 = (x_0^5, y_0^5)$,

where $x_0^5 = 0$ and $y_0^5 = 0$,

$E_1{}^5 = (x_1^5, y_1^5)$

where $x_1^5 = s_1^5$ and $y_1^5 = r_4\left(r_2 - \sqrt{r_2^2 + s_1^5 r_3\left(-s_1^5 r_1 + r_5\right)}\right)$.

and $E_2{}^5 = (x_2^5, y_2^5)$, where $x_2^5 = s_2^5$ and $y_2^5 = r_4\left(r_2 - \sqrt{r_2^2 + s_2^5 r_3\left(-s_2^5 r_1 + r_5\right)}\right)$,

$s_1^5 = \dfrac{s_1 - s_2}{s_3}$, $s_2^5 = \dfrac{s_1 + s_2}{s_3}$,

$$s_1 = r_3 r_4^2 r_5 \left(\varphi_1 - \varphi_4\right)^2 + 2\alpha_1 \left(\alpha_1 - \beta_1 + r_2 r_4 \left(-\varphi_1 + \varphi_4\right)\right)\psi_1 ,$$

$$r_1 = \alpha_1 \psi_1 ,\ r_2 = \alpha_2 - \beta_2 ,\ r_3 = 4\alpha_2 \psi_2 ,$$

$$r_4 = \frac{1}{2\alpha_2 \psi_2},\ r_5 = \alpha_1 - \beta_1 + \theta_1 ,$$

$$\beta_1 = \mu + \theta_1 + \theta_3 ,$$

$$\beta_2 = \mu + \theta_2 ,$$

$$\alpha_1 = \gamma_1 \pi ,$$

$s_2 = \sqrt{h}$ and $h = h_1 + h_2$,

$$h_1 = r_4^2 \left(\varphi_1 - \varphi_4\right)^2 \left(r_3^2 r_4^2 r_5^2 \left(\varphi_1 - \varphi_4\right)^2 - 4 r_1 r_3 \left(\alpha_1 - \beta_1\right)\right)\left(\alpha_1 - \beta_1 + 2 r_2 r_4 \left(-\varphi_1 + \varphi_4\right)\right),$$

$$h_2 = 4 r_3 r_5 \alpha_1 \left(\alpha_1 - \beta_1 + r_2 r_4 \left(-\varphi_1 + \varphi_4\right)\right)\psi_1 + 4 r_2^2 \alpha_1^2 \psi_1^2) ,$$

$$\alpha_2 = \gamma_2 \pi ,$$

$$s_3 = 2 r_1 r_3 r_4^2 \left(\varphi_1 - \varphi_4\right)^2 + 2\alpha_1^2 \psi_1^2 .$$

Case 7: Absence of public schools only

In absence of public schools the variables $x = 0$ while $y \neq 0$ and $z \neq 0$.

Therefore three equilibrium points are obtained namely; $E_0^{\ 6}(y_0^{\ 6}, z_0^{\ 6})$, where $y_0^{\ 6} = 0$ and $z_0^{\ 6} = 0$. Which corresponds to the trivial equilibrium points. $E_1^{\ 6}(y_1^{\ 6}, z_1^{\ 6})$ where,

$y_1^{\ 6} = \dfrac{p_1 + (-\mu + \alpha_2) p_2}{p_3}$ and

$$z_1^{\ 6} = \frac{(p_1 + p_2(-\mu + \alpha_2))(p_3(\alpha_1 - \beta_2 + \theta_2) - \alpha_1(p_1 + p_2(-\mu + \alpha_2))\psi_3)}{p_3^2(\mu - \alpha_2)} .$$

$$y_2^{\ 6} = \frac{p_1 + p_2(\mu - \alpha_2)}{p_3} ,$$

$$z_2{}^6 = \frac{(p_1 + p_2(\mu-\alpha_2))(p_3(\alpha_1-\beta_2+\theta_2)-\alpha_1(p_1+p_2(\mu-\alpha_2))\psi_3}{p_3^2(\mu-\alpha_2)}.$$

Where $\alpha_1 = \gamma_2\pi$, $\alpha_2 = (1-\gamma_1-\gamma_2)\pi$, $\beta_2 = \mu+\theta_2$,

$\psi_2 yz = \dfrac{\eta_2 yz}{B+y}$, $\psi_3 y = \dfrac{y}{k_2}$ $p_1 = (-\beta_2+\theta_2)\psi_2 + \alpha_1(\psi_2 + (\mu-\alpha_2)\psi_3)$

, $p_2 = \sqrt{\dfrac{(\alpha_1-\beta_2+\theta_2)^2\psi_2^2}{(\mu-\alpha_2)^2} + \dfrac{2\alpha_1(\alpha_1-\beta_2+\theta_2)\psi_2\psi_3}{\mu-\alpha_2} + \alpha_1^2\psi_3^2}$ and $p_3 = 2\alpha_1\psi_2\psi_3$.

Case 8: Coexistence in the system

$x \neq 0$, $y \neq 0$, $z \neq 0$.

For the existence of the three categories, the study established many equilibrium points including complex equilibrium points. However, complex equilibrium points are not included because the system is defined in the positive region and also they are not tractable. The equilibrium points are $E_0{}^7 = (x_0^7, y_0^7, z_0^7)$, $E_1^7 = (x_1^7, y_1^7, z_1^7)$, $E_2^7 = (x_2^7, y_2^7, z_2^7)$ where $x_0^7 = 0$, $y_0^7 = 0$ and $z_0^7 = 0$. This corresponds to trivial equilibrium point.

$$x_1^7 = 0,\ y_1^7 = \frac{1}{2}\left(-\frac{-\mu+p_1+\alpha_3}{\varphi_2} + \frac{\alpha_2-\beta_2+\theta_2}{\alpha_2\psi_2}\right) \text{and}$$

$$z_1^7 = \frac{-\left(\left(\beta_2+\theta_2\right)\varphi_2\right)+\alpha_2\left(\varphi_2+\left(-\mu+\alpha_3\right)\psi_2\right)+\alpha_2\psi_2 p_1}{2\varphi_2^2}.$$

$$x_2^7 = p_2\left(p_3 - \frac{\varphi_3\left(-\mu-p_4+\alpha_3+p_2p_3\varphi_3-p_2\theta_3\varphi_3\right)}{p_5}\right), y_2^7 = 0 \text{ and}$$

$$z_2^7 = \frac{-\mu-p_4+\alpha_3+p_2p_3\varphi_3-p_2\theta_3\varphi_3}{p_5}.$$

$$x_3^7 = p_2\left(p_3 - \frac{\varphi_3\left(-\mu-p_4+\alpha_3+p_2p_3\varphi_3+p_2\theta_3\varphi_3\right)}{p_5}\right), y_3^7 = 0 \text{ and}$$

$$z_3^7 = \frac{-\mu-p_4+\alpha_3+p_2p_3\varphi_3+p_2\theta_3\varphi_3}{p_5}$$

$$x_4^7=\frac{p_2\left(hp_3-(m+n)(\varphi_1-\varphi_4)\right)}{h}\ ,\ y_4^7=\frac{m+n}{h}\ \text{ and }\ z_4^7=0$$

$$x_5^7=\frac{p_{11}}{\varphi_6}-\frac{p_{22}+\dfrac{p_{66}\varphi_6}{p_{77}}-\sqrt{p_{33}+\left(p_{44}-\dfrac{p_{66}\varphi_6}{p_{77}}\right)^2}}{p_{55}}\,,$$

$$y_5^7=\frac{p_{22}+\dfrac{p_{66}\varphi_6}{p_{77}}-\sqrt{p_{33}+\left(p_{44}-\dfrac{p_{66}\varphi_6}{p_{77}}\right)^2}}{p_{55}}\ \text{ and }\ z_5^7=\frac{p_{66}}{p_{77}}$$

$$x_6^7=\frac{p_{11}}{\varphi_6}-\frac{p_{22}+\dfrac{p_{66}\varphi_6}{p_{77}}+\sqrt{p_{33}+\left(p_{44}-\dfrac{p_{66}\varphi_6}{p_{77}}\right)^2}}{p_{55}}\,,$$

$$y_6^7=\frac{p_{22}+\dfrac{p_{66}\varphi_6}{p_{77}}+\sqrt{p_{33}+\left(p_{44}-\dfrac{p_{66}\varphi_6}{p_{77}}\right)^2}}{p_{55}}\ \text{ and }\ z_6^7=\frac{p_{66}}{p_{77}}\ .$$

Where

$$\beta_1=\mu+\theta_1+\theta_3,$$

$$\beta_2=\mu+\theta_2,\ \ p_3=\alpha_1-\beta_1,$$

$$p_{11}=\mu-\alpha_3-\theta_2-\theta_3,$$

$$p_{22}=-\alpha_2+\beta_2+\theta_1-\frac{p_{11}\varphi_3}{\varphi_6}+\frac{p_{11}\varphi_5}{\varphi_6},$$

$$p_{33}=-\frac{4p_{11}\theta_1\left(-\alpha_2\varphi_2-\varphi_3+\varphi_5\right)}{\varphi_6},$$

$$p_{44}=\alpha_2-\beta_2-\theta_1+\frac{p_{11}\varphi_3}{\varphi_6}-\frac{p_{11}\varphi_5}{\varphi_6},$$

$$p_{55}=2\left(-\alpha_2\varphi_2-\varphi_3+\varphi_5\right),$$

$$p_{66}=-p_{11}^2p_{55}^2+2p_{11}p_{22}p_{55}\varphi_6-p_{22}^2\varphi_6^2+p_{33}\varphi_6^2+p_{44}^2\varphi_6^2,$$

$$p_{77}=2\varphi_6^2\left(-p_{11}p_{55}+p_{22}\varphi_6+p_{44}\varphi_6\right)$$

$$p_1=\sqrt{\frac{4\left(\alpha_2-\beta_2\right)\theta_2\varphi_2^2+\left(\left(-\alpha_2+\beta_2+\theta_2\right)\varphi_2+\alpha_2\left(\mu-\alpha_3\right)\psi_2\right)^2}{\alpha_2^2\psi_2^2}},$$

$$p_2=\frac{1}{\alpha_1\psi_1},\ p_5=2p_2\varphi_3^2$$

$$p_4=\sqrt{4p_2^2p_3\theta_3\varphi_3^2+\left(-\mu+\alpha_3+p_2p_3\varphi_3-p_2\theta_3\varphi_3\right)^2},$$

$$\gamma_1\pi=\alpha_1,\ \gamma_2\pi=\alpha_2,\ (1-\gamma_1-\gamma_2)\pi=\alpha_3,$$

$$\psi_1x=\frac{x}{k_1},\ \psi_2y=\frac{y}{k_2},\ \varphi_1xy=\frac{\eta_1xy}{B+x},$$

$$\varphi_2xy=\frac{\eta_3yz}{B+y},$$

$$\varphi_3xy=\frac{\eta_3xz}{B+x},$$

$$\varphi_4xy=\frac{\eta_3xy}{B+y},$$

$$b_1=b_{11}+b_{12}$$

$$b_2=\frac{b_{21}+b_{22}+b_{23}}{b_{24}},$$

$$n_1=\alpha_1\varphi_1,$$

$$n_2=\frac{1}{2\left(-n_1\alpha_2\varphi_2-\varphi_3^2+\varphi_3\varphi_6\right)}$$

$$,\ n_3=-4\left(\alpha_1\theta_1-\beta_1\theta_1-z\theta_1\varphi_5\right),$$

$$n_4 = -n_1\alpha_2 + n_1\beta_2 - \alpha_1\varphi_3 + \beta_1\varphi_3 + \theta_1\varphi_3 + \alpha_1\varphi_6 - \beta_1\varphi_6\,,$$

$$b_{11} = (-n_1^2 n_2\theta_2\varphi_4 + (\sqrt{-n_3\left(n_1\alpha_2\varphi_2 + \varphi_3^2 - \varphi_3\varphi_6\right)})\{n_2\varphi_3\varphi_5 - n_1 n_2\varphi_4\} +$$

$$\varphi_5\left(\alpha_1 - n_2 n_4\varphi_3 - \theta_3\left(1 + n_2\varphi_3\left(\varphi_3 - 2\varphi_6\right)\right) + \beta_1\left(-1 + n_2\varphi_3\varphi_6\right)\right)$$

$$b_{12} = \beta_1\left(-1 + n_2\varphi_3\varphi_6\right)) + n_1\left(-\mu + \alpha_3 + n_2\left(n_4\varphi_4 + \theta_3\varphi_3\varphi_4 + \theta_2\varphi_5\left(\varphi_3 - 2\varphi_6\right) - \beta_1\varphi_4\varphi_6\right)\right),$$

$$b_{21} = (n_1\sqrt{(\frac{1}{n_1^2}(4\left(n_1^2 n_2\varphi_4^2 + \varphi_5^2\left(1 + n_2\varphi_3\left(\varphi_3 - 2\varphi_6\right)\right) + 2n_1 n_2\varphi_4\varphi_5\left(-\varphi_3 + \varphi_6\right)\right)((\alpha_1 - \beta_1)\theta_3}$$

$$+n_2\left(-n_1\theta_2 + \theta_3\varphi_3\right)(-n_4 + \beta_1\varphi_6 + \sqrt{-n_3\left(n_1\alpha_2\varphi_2 + \varphi_3^2 - \varphi_3\varphi_6\right)})\,,$$

$$b_{22} = n_1^2 n_2\theta_2\varphi_4 + \left(-\alpha_1 + \beta_1\right)\varphi_5 + \varphi_5(\theta_3\left(1 + n_2\varphi_3\left(\varphi_3 - 2\varphi_6\right)\right) - n_2\varphi_3(-n_4 + \beta_1\varphi_6 +$$

$$+\sqrt{-n_3\left(n_1\alpha_2\varphi_2 + \varphi_3^2 - \varphi_3\varphi_6\right)})),$$

$$b_{23} = n_1(\mu - \alpha_3 + n_2(-n_4\varphi_4 - \theta_3\varphi_3\varphi_4 - \theta_2\varphi_5\left(\varphi_3 - 2\varphi_6\right) + \varphi_4(\beta_1\varphi_6 +$$

$$\sqrt{-n_3\left(n_1\alpha_2\varphi_2 + \varphi_3^2 - \varphi_3\varphi_6\right)}))))^2)),$$

$$b_{24} = \left(2\left(n_1^2 n_2\varphi_4^2 + \varphi_5^2\left(1 + n_2\varphi_3\left(\varphi_3 - 2\varphi_6\right)\right) + 2n_1 n_2\varphi_4\varphi_5\left(-\varphi_3 + \varphi_6\right)\right)\right),$$

$$m = \alpha_2 - \beta_2 + p_2 p_3\varphi_1 - p_2\theta_1\varphi_1 - p_2 p_3\varphi_4 + p_2\theta_1\varphi_4\,, h = 2p_2\left(\varphi_1 - \varphi_4\right)^2 + 2\alpha_2\psi_2\,,$$

$$n = \sqrt{\left(\alpha_2 - \beta_2 + p_2\left(p_3 - \theta_1\right)\left(\varphi_1 - \varphi_4\right)\right)^2 + 4p_2 p_3\theta_1\left(p_2\left(\varphi_1 - \varphi_4\right)^2 + \alpha_2\psi_2\right)}\ .$$

Given that $\varphi_1, \varphi_2,\ \varphi_3,\ \varphi_4,\ \varphi_5,\ \varphi_6 \geq 0$ and $B > 0$.

4.5 Stability analysis

4.5.1 Local stability

The local stability of a steady state means that if you put the system somewhere nearby the equilibrium point, then it will move itself to the equilibrium point after some time t.

The local stability of the system is determined by the nature of eigenvalues of the variational matrix, that is, for negative eigenvalues implies a stable system otherwise unstable. In case where the matrix is large such that the nature of eigenvalues cannot be directly determined, Routh-Hurwitz criteria can be applied to determine the stability.

The system has eight solutions;

$E^0(0,0,0), E^1(K_1,0,0), E^2(0,K_2,0), E^3(0,0,K_3), E^4(x^*,y^*,0), E^5(x^*,0,z^*), E^6(0,y^*,z^*)$

$, E^7(x^*,y^*,z^*)$

The variational matrix of the system is given by

$$\begin{pmatrix} b_{11} & -\frac{x\eta_1}{x+B_1}-\frac{xy\eta_4}{(y+B_4)^2}+\frac{x\eta_4}{y+B_4} & -\frac{x\eta_3}{x+B_3} \\ \frac{yB_1\eta_1}{(y+B_1)^2}-\frac{y\eta_4}{y+B_4} & b_{22} & -\frac{y\eta_2}{y+B_2} \\ \frac{xz\eta_3}{(x+B_3)^2}+\frac{z\eta_3}{x+B_3}+\theta_3 & -\frac{yz\eta_2}{(y+B_2)^2}+\frac{z\eta_2}{y+B_2}+\theta_2 & b_{33} \end{pmatrix}, \quad (9)$$

where,

$$b_{11} = -\mu + (\pi - \frac{2\pi x}{k_1})\gamma_1 - \frac{yB_1\eta_1}{(x+B_1)^2} - \frac{zB_3\eta_3}{(x+B_3)^2} + \frac{y\eta_4}{(y+B_4)} - \theta_1 - \theta_3,$$

$$b_{22} = -\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 - \frac{x\eta_1}{x+B_1} + \frac{zB_2\eta_2}{(y+B_2)^2} - \frac{xB_4\eta_4}{(y+B_4)^2} - \theta_2 \text{ and}$$

$$b_{33} = -\mu + \pi\alpha + \frac{y\eta_2}{y+B_2} + \frac{x\eta_3}{x+B_3}.$$

Theorem 1: The trivial steady states , E^0 , is stable if the eigenvalues of the variational matrix are all less than zero. Otherwise unstable.

Proof

We proved the above theorem by obtaining the variational matrix of equations (3.5.1), (3.5.2) and (3.5.3). The variational matrix is a partial derivative of equation 3.5.1, 3.5.2 and 3.5.2 with respect to x, y, z. This was done according the scenario at hand.

The variational matrix of (9) evaluated at E_0 is given by,

$$\begin{pmatrix} -\mu+\pi\gamma_1-\theta_1-\theta_3 & 0 & 0 \\ \theta_1 & -\mu+\pi\gamma_1-\theta_2 & 0 \\ \theta_3 & \theta_2 & -\mu+\pi\alpha \end{pmatrix}$$

By letting $\rho=\theta_1+\theta_3$ and $\alpha=(1-\gamma_1-\gamma_2)$,

eigenvalues are
$$\lambda_1=\pi\alpha-\mu,$$
$$\lambda_2=\pi\gamma_2-\theta_2-\mu,$$
$$\lambda_3=\pi\gamma_1-\rho-\mu.$$

The steady state $E^0(0,0,0)$ is stable if and only if $\pi\gamma_2<\theta_2+\mu$, $\pi\alpha<\mu$ and $\pi\gamma_1<\rho+\mu$. Otherwise unstable.

Theorem 2: Equilibrium point, E^1, at the point $x\neq 0$, $y=0$, $z=0$, by Routh- Hurwitz criterion is locally asymptotically stable if $A_1, A_3>0$ and $A_1A_2-A_0A_3>0$. Otherwise the system is unstable.

Proof

$x^0=0$ corresponds to trivial equilibrium point.

$$x^1=\frac{(k_1(\mu-\pi\gamma_1+\theta_1+\theta_3)}{\pi\gamma_1}.$$

The variational matrix is given by,

$$\begin{pmatrix} a_{11} & a_{12} & -a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where

$$a_{11}=-\mu+(\pi-\frac{2\pi x^1}{k_1})\gamma_1-\theta_1-\theta_3,$$

$$a_{12}=-\frac{x^1\eta_1}{x^1+B}+\frac{x^1\eta_4}{B},$$

$$a_{13} = -\frac{x^1\eta_3}{x^1 + B}, \quad a_{21} = \theta_1$$

$$a_{22} = -\mu + \pi\gamma_2 + \frac{x^1\eta_1}{x^1 + B} - \frac{x^1\eta_4}{B} - \theta_2 \ ,$$

$$a_{31} = \theta_3 \ , \ a_{32} = \theta_2 \text{ and } a_{33} = -\mu + \pi\alpha + \frac{x^1\eta_3}{x^1 + B}.$$

By getting the determinant of the below matrix resulted in the preceding equation

$$\begin{pmatrix} a_{11} - p & a_{12} & -a_{13} \\ a_{21} & a_{22} - p & 0 \\ a_{31} & a_{32} & a_{33} - p \end{pmatrix}$$

Implying that,

$$a_{11}a_{22}a_{33} - pa_{11}a_{22} - pa_{11}a_{33} + p^2a_{11} - pa_{22}a_{33} + p^2a_{22} + p^2a_{33} - p^3 - a_{13}a_{21}a_{32} - a_{12}a_{21}a_{33} + pa_{12}a_{21}$$

$$+a_{13}a_{31}a_{22} - pa_{13}a_{31} = 0.$$

By simplifying

$$-p^3 + p^2(a_{11} + a_{22} + a_{33}) - p(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} + a_{13}a_{31}) + a_{11}a_{22}a_{33} - a_{13}a_{21}a_{32}$$

$$-a_{12}a_{21}a_{33} + pa_{12}a_{21} + a_{13}a_{31}a_{22} - pa_{13}a_{31} = 0.$$

$$p^3 - p^2(a_{11} + a_{22} + a_{33}) + p(a_{11}a_{22} + a_{11}a_{33} + a_{22}a_{33} - a_{12}a_{21} + a_{13}a_{31}) - a_{11}a_{22}a_{33} + a_{13}a_{21}a_{32}$$

$$+a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22} = 0.$$

The characteristic equation is given by,

$p^3 + p^2A_1 + pA_2 + A_3 = 0$, where

$\mathrm{A}_1 = -(a_{11} + a_{22} + a_{33})$,

$A_2 = a_{11}a_{22} + a_{22}a_{33} + a_{11}a_{33} - a_{12}a_{21} + a_{13}a_{31}$ and

$A_3 = a_{13}a_{21}a_{32} - a_{11}a_{22}a_{33} + a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$.

Theorem 3: Equilibrium point, E^2, at the point $x=0$, $y \neq 0$, $z=0$ is stable if and only if the eigenvalues of the variational matrix are less than zero.

Proof

The variational matrix is given by,

$$\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

Where

$$a_{11} = -\mu + \pi\gamma_1 - \frac{y\eta_1}{B} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3,$$

$$a_{21} = \frac{y\eta_1}{B} - \frac{y\eta_4}{y+B} + \theta_1 \ ,$$

$$a_{22} = -\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 - \theta_2,$$

$$a_{23} = -\frac{y\eta_2}{y+B},$$

$$a_{31} = \theta_3 \ , \quad a_{32} = \theta_2, \quad a_{33} = -\mu + \pi\alpha + \frac{y\eta_2}{y+B}$$

The eigenvalues are $-\mu + \pi\gamma_1 - \frac{y\eta_1}{B} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3$, $-\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 - \theta_2$ and $-\mu + \pi\alpha + \frac{y\eta_2}{y+B}$.

The eigenvalues are stable if,

$$(-\mu + \pi\gamma_1 - \frac{y\eta_1}{B} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3) < 0,$$

$$(-\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 - \theta_2) < 0 \text{ and}$$

$$(-\mu + \pi\alpha + \frac{y\eta_2}{y+B}) < 0 .$$

Theorem 4: Equilibrium point, E^3, for point $x = 0$, $y = 0$, $z \neq 0$ is stable if and only if $\mu > \pi\alpha$, $\mu > \pi\gamma_2 + \theta_2$ and $\mu > \pi\gamma_1 + \rho$ otherwise unstable.

Proof: The variational matrix is given by,

$$\begin{pmatrix} -\mu + \pi\gamma_1 - \theta_1 - \theta_3 & 0 & 0 \\ \theta_1 & -\mu + \pi\gamma_1 - \theta_2 & 0 \\ \theta_3 & \theta_2 & -\mu + \pi\alpha \end{pmatrix}$$

By choosing $\rho = \theta_1 + \theta_3$ and $\alpha = (1 - \gamma_1 - \gamma_2)$, the eigenvalues are

$$\begin{aligned} \lambda_1 &= \pi\alpha - \mu, \\ \lambda_2 &= \pi\gamma_2 - \theta_2 - \mu, \\ \lambda_3 &= \pi\gamma_1 - \rho - \mu. \end{aligned}$$

The eigenvalues are negative if and only if $\mu > \pi\alpha$, $\mu > \pi\gamma_2 + \theta_2$ and $\mu > \pi\gamma_1 + \rho$ for it to be stable, otherwise unstable.

Theorem 5: Equilibrium point, E^4, for case where only $y = 0$ by Routh-Hurwitz criterion is stable if $\mathrm{D}_1, \mathrm{D}_3 > 0$ and $\mathrm{D}_1\mathrm{D}_2 > \mathrm{D}_0 D_3$, otherwise unstable.

Proof

The variational matrix of the steady states of E^4 is given by,

$$\begin{pmatrix} d_{11} & d_{12} & -d_{13} \\ d_{21} & d_{22} & 0 \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \text{where},$$

$$\mathrm{d}_{11} = -\mu + (\pi - \frac{2\pi x}{k_1})\gamma_1 - \frac{zB\eta_3}{(x+B)^2} - \theta_1 - \theta_3,$$

$$d_{12} = -\frac{x\eta_1}{x+B} + \frac{x\eta_4}{B},\ d_{13} = \frac{x\eta_3}{x+B},\ d_{21} = \theta_1,$$

$$d_{22} = -\mu + \pi\gamma_2 + \frac{x\eta_1}{x+B} - \frac{z\eta_4}{B} - \frac{x\eta_4}{B} - \theta_2,$$

$$d_{31} = -\frac{xz\eta_3}{(x+B)^2} + \frac{z\eta_3}{x+B} + \theta_3,$$

$$d_{32} = \frac{z\eta_2}{B} + \theta_2,$$

$$d_{33} = -\mu + \pi\alpha + \frac{x\eta_3}{x+B}.$$

By getting determinant of the below matrix resulted in the preceding equation below,

$$\begin{pmatrix} d_{11}-r & d_{12} & -d_{13} \\ d_{21} & d_{22}-r & 0 \\ d_{31} & d_{32} & d_{33}-r \end{pmatrix}$$

$$d_{11}d_{22}d_{33} - rd_{11}d_{22} - rd_{11}d_{33} + r^2d_{11} - rd_{22}d_{33} + r^2d_{22} + r^2d_{33} - r^3 - d_{13}d_{21}d_{32} - d_{12}d_{21}d_{33} + rd_{12}d_{21}$$

$$+d_{13}d_{31}d_{22} - rd_{13}d_{31} = 0.$$

By simplifying

$$-r^3 + r^2(d_{11} + d_{22} + d_{33}) - r(d_{11}d_{22} + d_{11}d_{33} + d_{22}d_{33} - d_{12}d_{21} + d_{13}d_{31}) + d_{11}d_{22}d_{33} - d_{13}d_{21}d_{32}$$

$$-d_{12}d_{21}d_{33} + d_{13}d_{31}d_{22} = 0.$$

Further simplification yields,

$$r^3 - r^2(d_{11} + d_{22} + d_{33}) + r(d_{11}d_{22} + d_{11}d_{33} + d_{22}d_{33} - d_{12}d_{21} + d_{13}d_{31}) - d_{11}d_{22}d_{33} + d_{13}d_{21}d_{32}$$

$$+d_{12}d_{21}d_{33} - d_{13}d_{31}d_{22} = 0$$

The characteristic equation was given by $r^3 + r^2 D_1 + rD_2 + D_3 = 0$, where

$\mathrm{D}_1 = -(d_{11} + d_{22} + d_{33})$, $D_2 = d_{11}d_{33} + d_{22}d_{33} + d_{11}d_{22} - d_{12}d_{21} + d_{13}d_{31}$,

$D_3 = d_{13}d_{21}d_{32} - d_{11}d_{22}d_{33} + d_{12}d_{21}d_{33} - d_{13}d_{31}d_{22}$.

Theorem 6: Equilibrium point, E^5,for the case where only $z = 0$ is stable if $C_1, C_3 > 0$ and $C_1C_2 > C_0C_3$, otherwise unstable.

Proof: The variational matrix for the steady states above is given by,

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & -c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}, \text{ where}$$

$$c_{11} = -\mu + \left(p - \frac{2\pi x}{k_1} \right)\gamma_1 - \frac{yB\eta_1}{(x+B)^2} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3,$$

$$c_{12} = -\frac{x\eta_1}{x+B} - \frac{xy\eta_4}{(y+B)^2} + \frac{x\eta_4}{y+B},$$

$$c_{13} = \frac{x\eta_3}{x+B}, \; c_{21} = \frac{yB\eta_1}{(x+B)^2} - \frac{y\eta_4}{y+B} + \theta_1,$$

$$c_{22} = -\mu + \left(\pi - \frac{2\pi y}{k_2} \right)\gamma_2 + \frac{x\eta_1}{x+B} - \frac{xB\eta_4}{(y+B)^2} - \theta_2,$$

$$c_{23} = \frac{y\eta_2}{y+B}, \; c_{31} = \theta_3, \; c_{32} = \theta_2,$$

$$c_{33} = -\mu + \pi\alpha + \frac{y\eta_2}{y+B} + \frac{x\eta_3}{x+B}.$$

By getting the determinant of the below matrix resulted to the preceding equation below

$$\begin{pmatrix} c_{11}-m & c_{12} & c_{13} \\ c_{21} & c_{22}-m & c_{23} \\ c_{31} & c_{32} & c_{33}-m \end{pmatrix}$$

$$c_{11}c_{22}c_{33}-mc_{11}c_{22}-mc_{11}c_{33}+m^2c_{11}-mc_{22}c_{33}+m^2c_{22}+c^2c_{33}-m^3-c_{12}c_{21}c_{33}+mc_{12}c_{21}+c_{23}c_{32}c_{11}$$

$$-mc_{23}c_{32}-c_{13}c_{31}c_{22}+mc_{13}c_{31}=0.$$

By simplifying the above resulted to,

$$-m^3+m^2(c_{11}+c_{22}+c_{33})-m(c_{11}c_{22}+c_{11}c_{33}+c_{22}c_{33}-c_{12}c_{21}+c_{23}c_{32}-c_{13}c_{31})+c_{11}c_{22}c_{33}-c_{12}c_{21}c_{33}$$

$$+c_{23}c_{32}c_{11}-c_{13}c_{31}c_{22}=0.$$

Implying that,

$$m^3-m^2(c_{11}+c_{22}+c_{33})+m(c_{11}c_{22}+c_{11}c_{33}+c_{22}c_{33}-c_{12}c_{21}+c_{23}c_{32}-c_{13}c_{31})+c_{12}c_{23}c_{31}-c_{11}c_{22}c_{33}$$

$$+c_{13}c_{21}c_{32}+c_{12}c_{21}c_{33}-c_{31}c_{22}c_{13}=0.$$

The characteristic equation is given by $m^3+m^2C_1+mC_2+C_3=0$, where $\mathrm{C}_1=-(c_{11}+c_{22}+c_{33})$, $C_2=c_{11}c_{33}+c_{22}c_{33}+c_{11}c_{22}-c_{12}c_{21}+c_{23}c_{32}+c_{13}c_{31}$ and

$C_3=c_{12}c_{23}c_{31}-c_{11}c_{22}c_{33}+c_{13}c_{21}c_{32}+c_{12}c_{21}c_{33}-c_{31}c_{22}c_{13}$.

Theorem 7: Equilibrium point , E^6, for $x=0$, $y\neq 0$, $z\neq 0$, by Routh-Hurwitz is stable if $G_1,G_3>0$ and $G_1G_2>G_0G_3$, otherwise unstable.

Proof

The variational matrix for the equilibrium point at $x=0$, $y\neq 0$, $z\neq 0$ is given by,

$$\begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{22} & -g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix},$$

where

$$g_{11} = -\mu + \pi\gamma_1 - \frac{y\eta_1}{B} - \frac{z\eta_3}{B} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3,$$

$$g_{22} = -\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 - \frac{zB\eta_2}{(y+B)^2} - \theta_2,$$

$$g_{23} = \frac{y\eta_2}{y+B}, \quad g_{31} = \frac{z\eta_3}{B} + \theta_3,$$

$$g_{32} = -\frac{yz\eta_2}{(y+B)^2} + \frac{z\eta_2}{y+B} + \theta_2 \text{ and}$$

$$g_{33} = -\mu + \pi\alpha + \frac{y\eta_2}{y+B}.$$

By getting the determinant of the matrix resulted in the preceding equation,

$$\begin{pmatrix} g_{11}-s & g_{12} & g_{13} \\ g_{21} & g_{22}-s & g_{23} \\ g_{31} & g_{32} & g_{33}-s \end{pmatrix}$$

$$g_{11}g_{22}g_{33} - sg_{11}g_{22} - sg_{11}g_{33} + s^2g_{11} - sg_{22}g_{33} + s^2g_{22} + s^2g_{33} - s^3 + g_{23}g_{32}g_{11} + sg_{23}g_{32} = 0.$$

By simplifying the above resulted to,

$$-s^3 + s^2(g_{11} + g_{22} + g_{33}) - s(g_{11}g_{22} + g_{11}g_{33} + g_{22}g_{33} + g_{23}g_{32}) + g_{11}g_{22}g_{33} + g_{23}g_{32}g_{11} = 0.$$

$$s^3 - s^2(g_{11} + g_{22} + g_{33}) + s(g_{11}g_{22} + g_{11}g_{33} + g_{22}g_{33} + g_{23}g_{32}) - (g_{11}g_{22}g_{33} + g_{23}g_{32}g_{11}) = 0.$$

The resulting characteristic equation is given by $s^3 + s^2G_1 + sG_2 + G_3 = 0$, where $G_1 = -(g_{11} + g_{22} + g_{33})$, $G_2 = g_{11}g_{33} + g_{22}g_{33} + g_{11}g_{22} + g_{23}g_{32}$ and $G_3 = -(g_{11}g_{22}g_{33} + g_{11}g_{23}g_{32})$.

Theorem 8: Equilibrium point, E^7 ,for the case $x \neq 0$, $y \neq 0$, $z \neq 0$ is stable if $K_1, K_3 > 0$ and $K_1 K_2 > K_3 K_0$, otherwise unstable.

Proof : The variational matrix for the equilibrium point at $x \neq 0$, $y \neq 0$, $z \neq 0$ is given by,

$$\begin{pmatrix} k_{11} & k_{12} & -k_{13} \\ k_{21} & k_{22} & -k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}, \text{ where}$$

$$k_{11} = -\mu + (\pi - \frac{2\pi x}{K_1})\gamma_1 - \frac{yB\eta_1}{(x+B)^2} - \frac{zB\eta_3}{(x+B)^2} + \frac{y\eta_4}{y+B} - \theta_1 - \theta_3,$$

$$k_{12} = -\frac{x\eta_1}{(x+B)^2} - \frac{xy\eta_4}{(y+B)^2} + \frac{x\eta_4}{y+B},$$

$$k_{13} = \frac{x\eta_3}{x+B} \quad , k_{21} = -\frac{yB\eta_4}{(x+B)^2} - \frac{y\eta_4}{y+B} + \theta_1,$$

$$k_{31} = -\frac{zx\eta_3}{(x+B)^2} + \frac{z\eta_3}{x+B} + \theta_3, \qquad k_{23} = \frac{y\eta_2}{y+B},$$

$$k_{22} = -\mu + (\pi - \frac{2\pi y}{k_2})\gamma_2 + \frac{x\eta_1}{(x+B)} - \frac{zB\eta_2}{(y+B)^2} - \frac{xB\eta_4}{(y+B)^2} - \theta_2,$$

$$k_{32} = -\frac{zy\eta_2}{(y+B)^2} + \frac{z\eta_2}{y+B} + \theta_2 \text{ and}$$

$$k_{33} = -\mu + \pi\alpha + \frac{y\eta_2}{y+B} + \frac{x\eta_3}{x+B}.$$

By getting the determinant of the matrix resulted in the preceding equation,

$$\begin{pmatrix} k_{11} - r & k_{12} & -k_{13} \\ k_{21} & k_{22} - r & -k_{23} \\ k_{31} & k_{32} & k_{33} - r \end{pmatrix}$$

$k_{11}k_{22}k_{33} - rk_{11}k_{22} - rk_{11}k_{33} + r^2k_{11} - rk_{22}k_{33} + r^2k_{22} + r^2k_{33} - r^3 - k_{12}k_{23}k_{31} - k_{13}k_{21}k_{32} - k_{12}k_{21}k_{33} + rk_{12}k_{21}$

$+k_{23}k_{32}k_{11} - rk_{23}k_{32} + k_{13}k_{31}k_{22} - rk_{13}k_{31}=0.$

By simplifying above resulted to,

$-r^3 + r^2(k_{11} + k_{22} + k_{33}) - r(k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}k_{21} + k_{23}k_{32} + k_{13}k_{31}) + k_{11}k_{22}k_{33} - k_{12}k_{23}k_{31}$

$-k_{13}k_{21}k_{32} - k_{12}k_{21}k_{33} + k_{23}k_{32}k_{11} + k_{13}k_{31}k_{22} = 0.$

$r^3 - r^2(k_{11} + k_{22} + k_{33}) + r(k_{11}k_{22} + k_{11}k_{33} + k_{22}k_{33} - k_{12}k_{21} + k_{23}k_{32} + k_{13}k_{31}) + k_{12}k_{23}k_{31} - k_{11}k_{22}k_{33}$

$+k_{13}k_{21}k_{32} + k_{12}k_{21}k_{33} - k_{23}k_{32}k_{11} - k_{13}k_{31}k_{22} = 0.$

The resulting characteristic equation is given by,

$r^3 + r^2K_1 + rK_2 + K_3 = 0$,where, $K_1 = -(k_{11} + k_{22} + k_{33})$

$K_2 = k_{11}k_{33} + k_{22}k_{33} + k_{11}k_{22} - k_{12}k_{21} + k_{23}k_{32} + k_{13}k_{31}$ and

$K_3 = k_{12}k_{23}k_{31} - k_{11}k_{22}k_{33} + k_{13}k_{21}k_{32} + k_{12}k_{21}k_{33} - k_{23}k_{32}k_{11} - k_{13}k_{31}k_{22}.$

4.5.2 Global stability

A global stability is the stability of a steady state at a point away from the equilibrium point. One common approach in studying the global asymptotically stable state is by use of Lyapunov function. Linear stability analysis tells us how a system behaves near an equilibrium point. It does not however tell us anything about what happens farther away from equilibrium (Keya & Islam, 2020). For higher dimensions, we considered using a Lyapunov function. This study considered Lyapunov similar to the one proposed by (Panja & Kumar, 2015).

Theorem 9

Let

$$V = \frac{(X - X^*)^2}{2} + \delta_1 \frac{(Y - Y^*)^2}{2} + \delta_2 \frac{(Z - Z^*)^2}{2}. \tag{10}$$

be the Lyapunov function, where $\delta_1, \delta_2 > 0$ were taken properly such that the above function satisfied the Lyapunov conditions below.

$V'(E^*) = 0$ where $E^* = X^*, Y^*, Z^*$ and $V(X,Y,Z) > 0$. The time derivative of V is $\frac{dV}{dt} \le 0$ implied that the system is just stable and $\frac{dV}{dt} < 0$ implies that E^* is globally asymptotically stable.

Proof

$$V = \frac{(X - X^*)^2}{2} + \delta_1 \frac{(Y - Y^*)^2}{2} + \delta_2 \frac{(Z - Z^*)^2}{2} \quad . \tag{11}$$

Getting time derivative of equation 11,

$$\frac{dV}{dt} = (X - X^*)\frac{dX}{dt} + \delta_1 (Y - Y^*)\frac{dY}{dt} + \delta_2 (Z - Z^*)\frac{dZ}{dt} .$$

By substituting the values of $\frac{dX}{dt}, \frac{dY}{dt}, \frac{dZ}{dt}$ results to,

$$\frac{dV}{dt} = [(X - X^*)(\gamma_1 \pi X(1 - \frac{X}{k_1}) - (\mu + \theta_1 + \theta_3)X - \frac{\eta_1 XY}{B+X} - \frac{\eta_3 XZ}{B+X} + \frac{\eta_4 XY}{B+Y}$$

$$+\delta_1 (Y - Y^*)(\gamma_2 \pi Y(1 - \frac{Y}{k_2}) - \ (\mu + \theta_2)Y + \frac{\eta_1 XY}{B+X} - \frac{\eta_2 YZ}{B+X} + \theta_1 X - \frac{\eta_4 XY}{B+Y}$$

$$+\delta_2 (Z - Z^*)(1 - \gamma_1 - \gamma_2)\pi Z + \theta_3 X + \theta_2 Y + \frac{\eta_3 XZ}{B+X} + \frac{\eta_2 YZ}{B+Y} - \mu Z].$$

By factorizing $X, Y \ and \ Z$,

$$\frac{dV}{dt} = [(X - X^*)\{\gamma_1 \pi (1 - \frac{X}{k_1}) - (\mu + \theta_1 + \theta_3) - \frac{\eta_1 Y}{B+X} - \frac{\eta_3 Z}{B+X} + \frac{\eta_4 Y}{B+Y})(X)\} + \delta_1 (Y - Y^*)\{(\gamma_2 \pi (1 - \frac{Y}{k_2})$$

$$- \ (\mu + \theta_2) + \frac{\eta_1 X}{B+X} - \frac{\eta_2 Z}{B+X} + \theta_1 \frac{X}{Y} - \frac{\eta_4 X}{B+Y})(Y)\} + \delta_2 (Z - Z^*)\{(1 - \gamma_1 - \gamma_2)\pi + \theta_3 \frac{X}{Z} + \theta_2 \frac{Y}{Z}$$

$$+ \frac{\eta_3 X}{B+X} + \frac{\eta_2 Y}{B+Y} - \mu)(Z)\}].$$

At special case $X = X^*$, $Y = Y^*$ and $Z = Z^*$ implying that $X - X^*$, $Y - Y^*$ and $Z - Z^*$ are also equilibrium points, then substituting resulted to,

$$\frac{dV}{dt} = \Big[(X - X^*)\Big\{\gamma_1\pi(1 - \frac{X}{k_1}) - (\mu + \theta_1 + \theta_3) - \frac{\eta_1 Y}{B + X} - \frac{\eta_3 Z}{B + X} + \frac{\eta_4 Y}{B + Y})(X - X^*)\Big\}$$

$$+\delta_1(Y - Y^*)\Big\{(\gamma_2\pi(1 - \frac{Y}{k_2}) - \ (\mu + \theta_2) + \frac{\eta_1 X}{B + X} - \frac{\eta_2 Z}{B + X} + \theta_1\frac{X}{Y} - \frac{\eta_4 X}{B + Y})(Y - Y^*)\Big\}$$

$$+\delta_2(Z - Z^*)\Big\{(1 - \gamma_1 - \gamma_2)\pi + \theta_3\frac{X}{Z} + \theta_2\frac{Y}{Z} + \frac{\eta_3 X}{B + X} + \frac{\eta_2 Y}{B + Y} - \mu)(Z - Z^*)\Big\}\Big]. \qquad (12)$$

Further simplifying equation 12 resulted to,

$$\frac{dV}{dt} = -\Big[(X - X^*)^2\Big\{\gamma_1\pi(-1 + \frac{X}{k_1}) - (\mu + \theta_1 + \theta_3) + \frac{\eta_1 Y}{B + X} + \frac{\eta_3 Z}{B + X} - \frac{\eta_4 Y}{B + Y})\Big\}$$

$$-\delta_1(Y - Y^*)^2\Big\{(\gamma_2\pi(-1 + \frac{Y}{k_2}) + (\mu + \theta_2) - \frac{\eta_1 X}{B + X} - \frac{\eta_2 Z}{B + X} - \theta_1\frac{X}{Y} + \frac{\eta_4 X}{B + Y})\Big\}$$

$$-\delta_2(Z - Z^*)^2\Big\{(-1 + \gamma_1 + \gamma_2)\pi - \theta_3\frac{X}{Z} - \theta_2\frac{Y}{Z} - \frac{\eta_3 X}{B + X} - \frac{\eta_2 Y}{B + Y} + \mu)\Big\}\Big]. \qquad (13)$$

Since $(X - X^*)^2 = 0$ was the same as $X^2 - XX^* - XX^* + X^{*2} = 0$.

Implying $X(X - X^*) = X^*(X - X^*)$.Thus $X = X^*$. Similarly $Y = Y^*$ and $Z = Z^*$. Since $\delta_1 > 0$ and $\delta_2 > 0$, they were chosen such that $\frac{dV}{dt} \le 0$ and that the steady state E^* was globally asymptotically stable.

4.6 Bifurcation analysis

Lotka-Volterra system uses quite a number of parameters to describe and formulate the system. But if parameters used in the model are changed, other types of dynamical behavior may occur and the critical parameter values at which such transitions happen are called bifurcation points (Panja & Kumar, 2015). This study addressed the transition factors of the prey-predator interactions. This transition with respect to small changes is called Hopf bifurcation. Hopf bifurcation occurred at the point where the system had nonhyperbolic equilibrium point and has purely imaginary eigenvalues (Panja & Kumar, 2015).

This study considered μ, θ as the bifurcation parameters and μ^*, θ^* as the turning point for values of the bifurcation parameters.

Theorem 10

Positive equilibrium over real domain $\mathbb{R}$ was considered. Let $\Phi:(0,\infty)\to\mathbb{R}$ be the continuous differential function of μ. Then, $\Phi(\mu)=M_1(\mu)M_2(\mu)-M_3(\mu)$. Since μ^* is the positive root of $\Phi(\mu)=0$.

Thus Hopf bifurcation of inner equilibrium $E(X^*,Y^*,Z^*)$ occurred at $\mu=\mu^*$ if and only if ;

i) $\Phi(\mu^*)=0$

ii) $R_2(\mu^*)R_4(\mu^*)+R_1(\mu^*)R_3(\mu^*)\neq 0$.

Proof:

Considering the first condition above that $\Phi(\mu)=0$, the characteristic equation of the variational matrix in theorem 8 can be written as $(r^2+k_2)(r+k_1)=0$

The roots of the equation are $\alpha_1=-k_1, \alpha_2=-i\sqrt{k_2}, \alpha_3=i\sqrt{k_3}$.

There exist an interval of $\mu^*+\xi,\ \mu^*-\xi$ since $\Phi(\mu^*)$ is continuous of all its roots.

General form for the complex roots is given by,

$\theta_1(\mu)=\omega(\mu)+i\xi(\mu)$ and $\theta_2(\mu)=\omega(\mu)-i\xi(\mu)$. (14)

We check the consistency condition,

$$\frac{d\,\mathrm{Re}\,\theta_j}{d\mu}\neq 0;\ j=1,2.$$

By substituting the general form of the complex roots into the characteristic equation and getting the derivative results to,

$$\omega^3 - \omega^2 i\xi - 3\xi^2\omega + i\xi^3 + \sigma_1\omega^2 - 2i\xi\omega\sigma_1 - \sigma_1\xi^2 + \sigma_2\omega - i\xi\sigma_2 + \sigma_3 . \tag{15}$$

By implicit differentiation of equation 15 we get,

$$(3\omega^2 - 3\xi^2 + 2\sigma_1\omega + \sigma_2)\omega'(\mu) - (6\omega\xi + 2\sigma_1\xi)\xi'(\mu) + \sigma_1'\omega - \sigma_1'\xi^2 + \sigma_2'\omega + \sigma_2' .$$

By letting,

$$R_1(\mu) = 3\omega^2 - 3\xi^2 + 2\sigma_1\omega + \sigma_2 ,$$

$$R_2(\mu) = 6\omega\xi + 2\sigma_1\xi ,$$

$R_3 = \sigma_1'\omega - \sigma_1'\xi^2 + \sigma_2'\omega + \sigma_2'$ and

$$R_4 = 2\sigma_1'\omega\xi \ + \ \sigma_2'\xi$$

Thus we have,

$$R_1(\mu)\omega'(\mu) - R_2(\mu)\xi'(\mu) + R_3(\mu) = 0. \tag{16}$$

$$R_2(\mu)\omega'(\mu) + R_1(\mu)\xi'(\mu) + R_4(\mu) = 0 . \tag{17}$$

For $\xi'(\mu)$ at $\mu = \mu^*$, therefore $\dfrac{d\,\mathrm{Re}\,\theta_j}{d\mu} = \chi'(\mu^*)$.

By comparing the values of $\xi'(\mu)$ using equation 16 and 17.

$$\frac{R_1(\mu^*) + R_3(\mu^*)}{R_2(\mu^*)} = -\frac{R_2(\mu^*) + R_4(\mu^*)}{R_1(\mu^*)} . \tag{18}$$

$= -\dfrac{R_2(\mu^*)R_4(\mu^*) + R_1(\mu^*)R_3(\mu^*)}{R_1^2(\mu^*) + R_2^2(\mu^*)} \neq 0$, only if

$$R_2(\mu^*)R_4(\mu^*) + R_1(\mu^*)R_3(\mu^*) \neq 0 .$$

Thus consistency requirement was fulfilled and thus Hopf bifurcation was realized where $\mu = \mu^*$.

4.7 Numerical simulation

Table 3: Parameter description and values

Parameter	Description	Value	Source
π	Schools recruitment	4,583,396	(Ministry of Education, 2019)
γ_1	Enrolment rate in public schools	0.6976	(Ministry of Education, 2019)
γ_2	Enrolment rate in private schools	0.04988	(Ministry of Education, 2019)
k_1	Public schools carrying capacity	3,546,153	(Ministry of Education, 2019)
k_2	Public schools carrying capacity	297,114	(Ministry of Education, 2019)
θ_1	Transfer rate from public to private schools	0.0009	Estimated
θ_2	Transfer rate from private to non-enrolled	0.0015	(Kingdom *et al.*, 2015)
θ_3	Transfer rate from public to non-enrolled	0.0015	(Kingdom *et al.*, 2015)
μ	Exit rate due to completion	0.25	Estimated
η_1	Private school predation rate on public schools	0.0004	Estimated

η_2	Non-enrolled entity predation rate on private schools	0.0001	(Agmour *et al.*, 2018)
η_3	Non-enrolled entity predation rate on public schools	0.0008	(Agmour *et al.*, 2018)
η_4	Public school predation rate on private schools	0.0006	Estimated
B	Saturation constant	1	(Rosenbaum, n.d.)

In this section, the numerical analysis is done with the help of Wolfram Mathematica by considering the parameters values of the model cited in the table 1 above. Due to difficulty in parameter estimation, the study relied heavily on previous published work related to this study. The initial values of the variables used to generate graphs in figure 2 to 6 are $x = 3045227$, $y = 217724$ (Ministry of Education, 2019) in page 10 and since the data for the non-enrolled was unavailable, the study considered the school going age and further extracted the data from the 2019 national census statistics data. From the numerical analysis, the study found out that all the eight equilibrium points were unstable. Since our model lies in the real domain, imaginary equilibrium points were not verified.

For Theorem 1, the eigenvalues are $1.0884, -0.0315$ and 2.77264 hence it is unstable for theorem 1 because of positive eigenvalue present.

For theorem 2, the steady states are -3.2503. Now we have $A_1 = -13.2227 < 0$, $A_2 = 42.9696$, $A_3 = 1.2559$ thus unstable equilibrium point by Routh Hurwitz criterion.

Theorem 3, the equilibrium point -0.0425367, the eigenvalues are 2.7726, 0.0315 and 1.0888, hence theorem 3 is unstable since it has positive eigenvalues.

Theorem 4, the eigenvalues are 1.0884, -0.0315 and 2.77264 thus confirms that theorem 4 is unstable.

Theorem 5, The first set equilibrium points are $x_1^4 = 3.2503$ and $z_1^4 = -0.1143$ which unstable since $D_1 = 1.7168$, $D_2 = -2.9637$ and $D_3 = -0.0999 < 0$.

The second set of equilibrium points are $x_2^4 = 212.3727$ and $z_2^4 = 930083.112$; results to $D_1 = 188.24$, $D_2 = -205880$, $D_3 = 224010$. Thus theorem 5 is unstable.

Theorem 6, the equilibrium points are $x_1^5 = -3.8359$ and $y_1^5 = -0.0155 - 4.783\mathrm{i}$. Results to $C_1 = -10.4076 - 9.6953i$, $C_2 = 10.1555 + 100.899i$ and $C_3 = -0.005163 - 98.4232i$.

The second set of equilibrium points are $x_2^5 = 10.3364$ and $y_2^5 = -0.0155 - 7.852\mathrm{i}$. Thus $C_1 = 13.772 - 15.9126i$, $C_2 = -16.1924 - 219.157i$ and $C_3 = 0.009540 + 257.578i$. From the above it confirms that the system is unstable for theorem 6.

Theorem 7, the equilibrium points are $y_1^6 = -7.763 \times 10^{-13}$ and $z_1^6 = 1.97959 \times 10^{-12}$. By Routh-Hurwitz we have $G_1 = -3.8296$, $G_2 = 2.8963$ and $G_3 = 0.09505$. Since $G_3 < 0$, implies the system is unstable.

$y_2^6 = -13255.0965$ and $z_2^6 = 2.2493 \times 10^9$. We have $G_1 = 1.77256 \times 10^6$, $G_2 = -4.8339 \times 10^{10}$ and $G_3 = 5.26135 \times 10^{10}$. Since $G_1 G_2 - G_3 < 0$, the equilibrium points are unstable.

Theorem 8, the first set of equilibrium points are $x_1^7 = 0$, $y_1^7 = -1656.91$, $z_1^7 = 1867770.3$. By Routh-Hurwitz criterion we have $K_1 = -964.05$, $K_2 = -3656941.28$ and $K_3 = 3981178.6$. Since $K_1 < 0$, the equilibrium points are unstable.

The second set of equilibrium points are $x_2^7 = 3.25027,\ y_2^7 = 0,\ z_2^7 = -0.004477$. Thus $K_1 = 1.08835$, $K_2 = -2.92308$ and $K_3 = -0.0987$. Hence the system for the second set of equilibrium is unstable.

The third set of equilibrium point are $x_3^7 = 3.24852,\ y_3^7 = 0,\ z_3^7 = 7.98893$. Making $K_1 = 1.71558$, $K_2 = -2.957$ and $K_3 = -0.1023$. $x_4^7 = 3.25029,\ y_4^7 = 0.04497,\ z_4^7 = 0$. We have $K_1 = 1.08897$, $K_2 = -2.85285$ and $K_3 = -0.2985$. Hence the system for the third set of equilibrium is unstable.

The fourth set of equilibrium point are $x_5^7 = -13291.8,\ y_5^7 = -0.001216,\ z_5^7 = 1.1987 \times 10^8$ Thus $K_1 = 76202.2$, $K_2 = -2.7233 \times 10^8$ and $K_3 = 2.9643 \times 10^8$. Hence unstable.

The fifth set of equilibrium point are $x_6^7 = 1.8189 \times 10^{-12}$, $y_6^7 = -13291.8$ and $z_6^7 = 1.1987 \times 10^8$. Thus $K_1 = -10672.8$, $K_2 = -2.7233 \times 10^8$ and $K_3 = 2.9643 \times 10^8$. Hence by Routh-Hurwitz criterion theorem 8 is unstable.

4.8 Numerical results

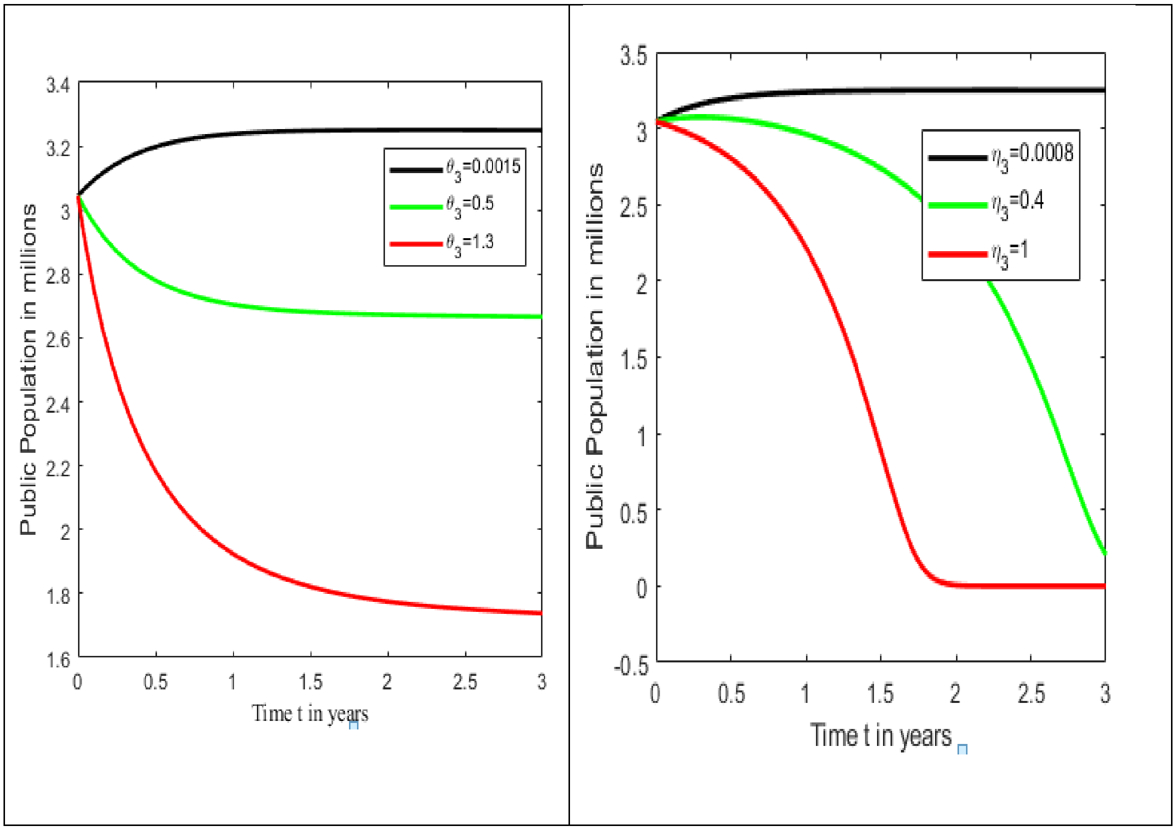

Figure 2: Dynamics of public schools with respect to θ_3 and η_3

Figure 2 illustrated the dynamics of public schools when varied with different values of θ_3 and η_3. In the first diagram, the value is varied using 0.0015, 0.5 and 1.3. From the diagram it is seen that an increase in the value of the (transfer rate from public schools to non-enrolled entity due to unknown factors), parameter θ_3 the population of public schools decrease steadily. An indication that the parameter is harmful to the public schools' population.

The second part of figure 2 also pointed out the change of public schools' population with respect to change in the value of the parameter η_3 (Non-enrolled entity predation rate on public schools). Increase in the value of η_3 caused a corresponding decrease in the population. If the value of η_3 is more than one, the population of public schools can become extinct.

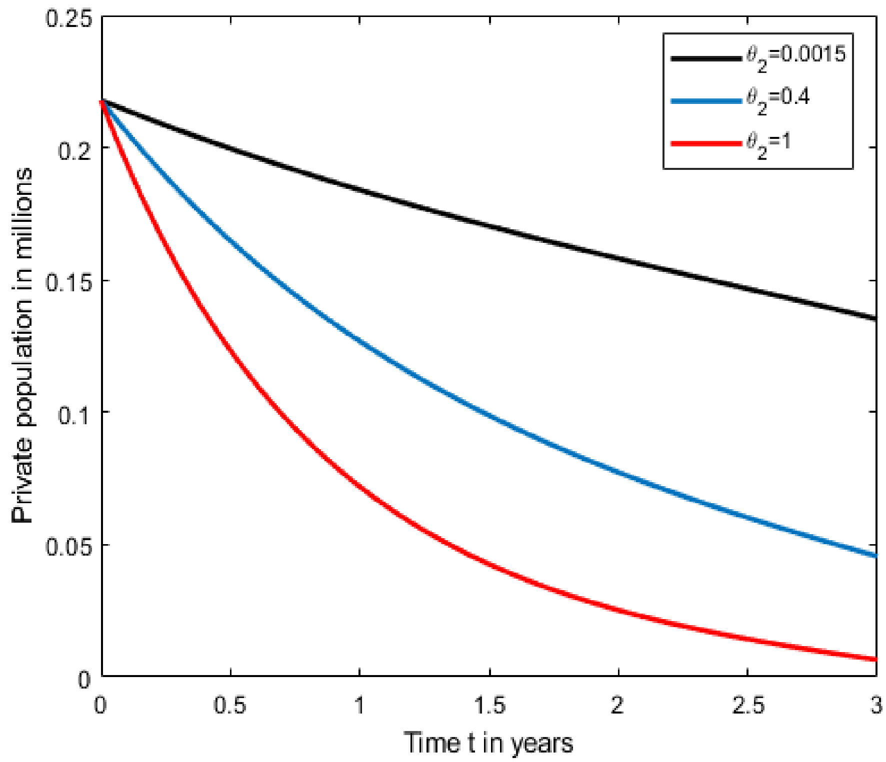

Figure 3:Dynamics of private schools with respect to θ_2

Figure 3 shows how private schools' population changes with the corresponding change in the value of the transfer rate from private to non-enrolled entity population. It clearly shows how the non-enrolled entity predates the private school population because the increase in the value of θ_2 cause a steady decrease in the population of private schools. This is an indicator of the parameter that causes change in the private school population.

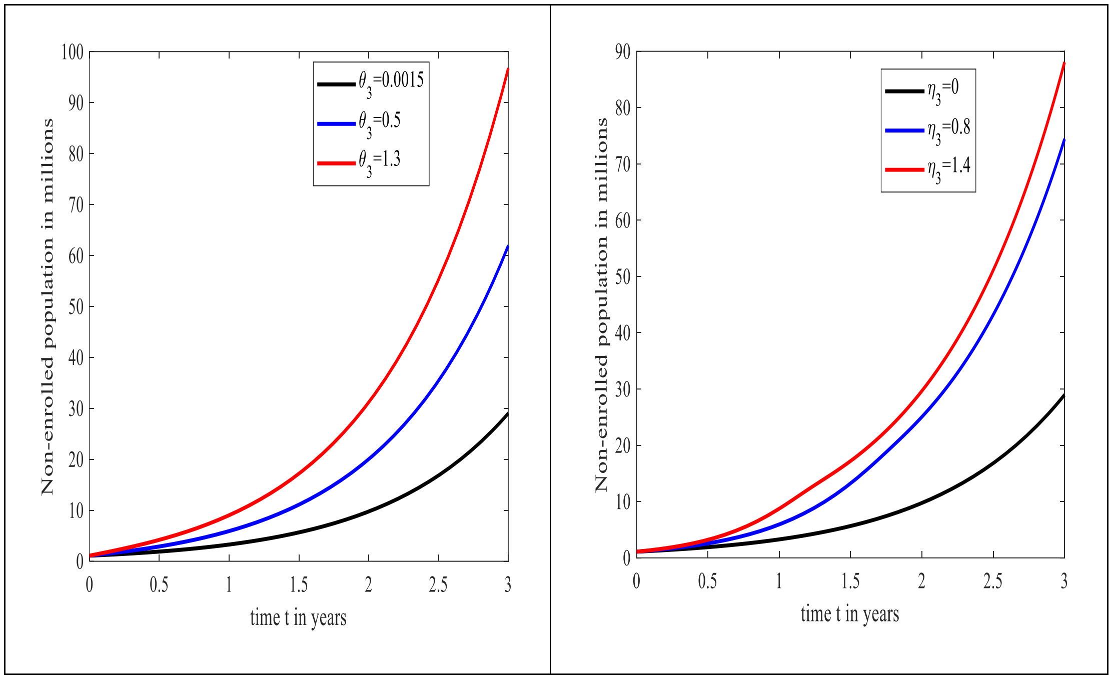

Figure 4: Dynamics of non-enrolled with respect to θ_3 and η_3

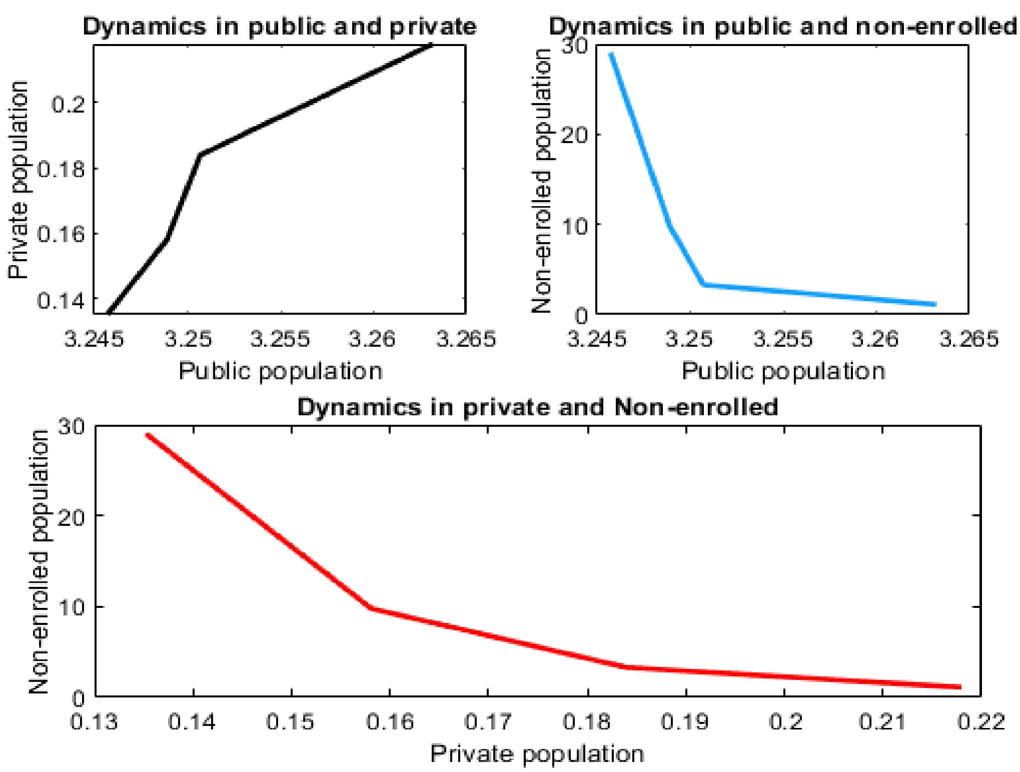

Figure 5: Dynamics of population (in millions) against each other at the same time interval

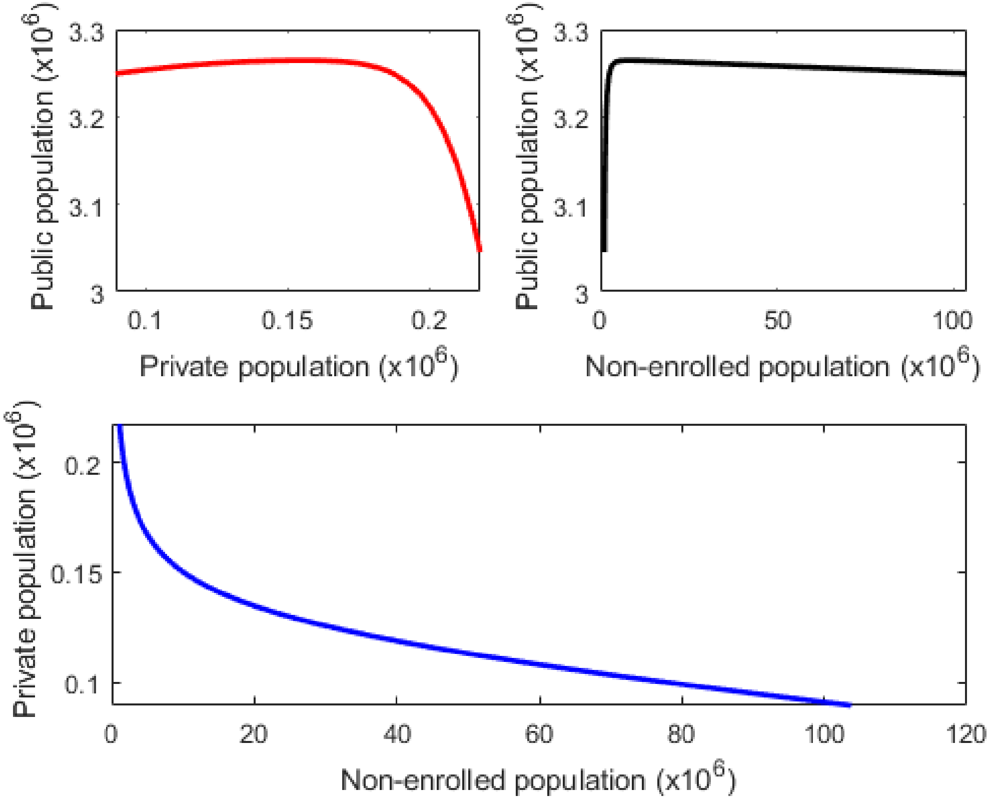

Figure 6: Population projection in the long run

In figure 6 showed the plotted two categories of schools. The first part of figure shows public schools against private schools. At first the population private and public schools increase simultaneously but at some point there was a sharp decrease in the population of private schools causing public schools population to increase gradually. The second part of figure 6 showed the graph of public schools' population plotted against non-enrolled entity. Public schools' population increased steadily while non-enrolled entity population remains constant at zero but at reached a point where public schools population remains constant as the non-enrolled entity population continued to increase. The last graph in figure 6 above private school population is plotted against non-enrolled entity. From the graph it is evident that the two categories the growth was exponential. Private schools' population increase caused a decrease in the population of the non-enrolled entity.

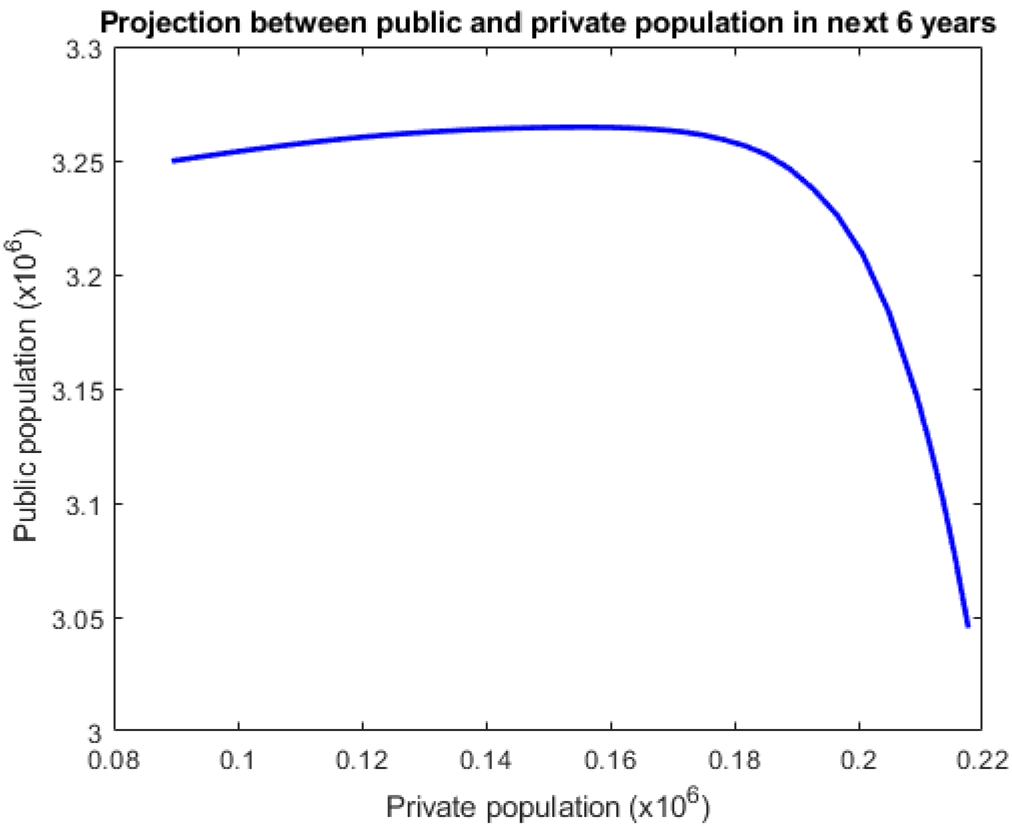

Figure 7: Public and private population in absence of non-enrolled entity

Plotting public primary schools' population against private primary schools' population showed that the growth of the population of both increases together but it reaches a point where the two schools' population gradually decreases.

4.9 Numerical bifurcation analysis

The study considered the resultant behavioral of the varied parameter with regard to the corrupt individuals in the future. The interest was in the critical values at which the asymptotic behavior changes qualitatively when the critical values are passed (Sooknanan *et al.*, 2016). This study had four steady states that may emerge a new steady states due to bifurcation of the system. Bifurcation points were found for μ as in figures 8.

In order to determine the direction of the bifurcation and to prove the stability, the study used the bifurcation theory discussed under section 4.6.

Bifurcation analysis was done for the positive equilibrium because the model under the study lies in the feasible region. The following plot were obtained using MATLAB.

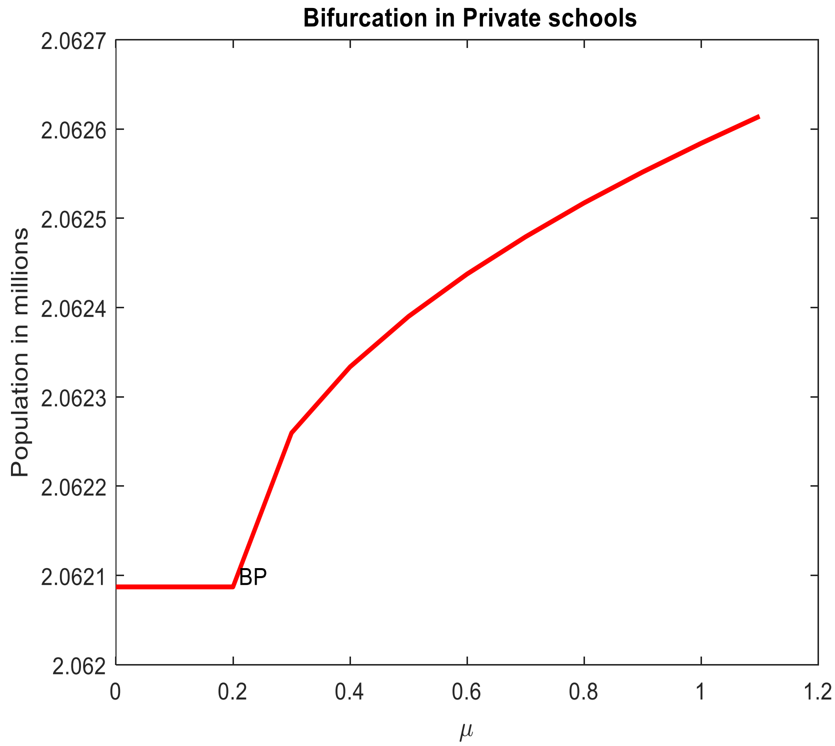

Figure 8: Bifurcation diagram for μ

In figure 8 bifurcation was done exit rate due to completion of the school study period as represented by parameter μ. From the varying of parameter μ, numerical simulation revealed that the population of both entities coexisted up to where the value of the is 0.2.

CHAPTER FIVE

DISCUSSION, CONCLUSION AND RECOMMENDATION

5.1 Discussion

Just like disease, predation helps in population regulation (Sooknanan *et al*., 2016). But in this study, predation from non-enrolled entity needs to be avoided at all cost to save the schools population, by avoiding school dropouts from learners. In this study the system of ODEs formed had eight equilibrium points. From the numerical simulation, all the eight equilibrium points were unstable. In figure 2, the parameter θ_3 and η_3 are varied from 0.0015 to 1.3 and 0.0008 to 1 respectively. In both cases as the value of the parameter increase the population of the public school and private school decreases. When the value of η_3 is 1, the population of private school becomes extinct. This implies these parameters are the ones responsible for the decrease in schools' population. In figure 4 as θ_3 and η_3 increases the population of the non-enrolled entity increases. This is because θ_3 and η_3 are the parameters that lead to predation of schools population as seen in figure 2. These parameters further cause a decrease in the private population as seen in figure 3. In figure 5, public and private schools' population are almost directly proportional to each other while for public and the non-enrolled, increases in the non-enrolled causes a decrease in the public population. This is due to predation of the non-enrolled as confirmed by figure 4. The same situation happens in the when private schools are plotted against the non-enrolled entity as seen in figure 5. Figure 6 shows population comparison in the long run. The first diagram shows the comparison between private schools' population and public schools population. Public and private schools' population increases for some time but starts to decrease steadily. For the comparison between public population and the non-enrolled category, there is a sharp increase in the population of public school while the non-enrolled remains constant but suddenly the population of public schools remains while the non-enrolled entity population increases with time. Finally, in the comparison between private and the non-enrolled entity population, the study found out an increased and a decreased population respectively. For figure 7 showed behavior of public and private schools in the long in the absence of the non-enrolled. Thus the plot diagram between public schools' population and private schools' population in the next six years. The two categories increased but starts to decrease steadily.

Figure 8 presented bifurcation diagram with respect to μ. Bifurcation diagram confirmed the theoretical results of theorem 8. The system of schools and the non-enrolled entity tended to coexistence when $\mu > 0.2$ as had been illustrated in theorem 8 ,the coexistence of the system.

In comparison with the previous research done on schools as highlighted by (Talance, 2020), this study is more suitable for the education system because of its ability to predict the future occurrence in terms of transfer, future population and the possible outcomes as indicated by parameter θ_3 and η_3. This is not possible by the previous research as the previously conducted studies concentrated only on present scenarios (McGee *et al.*, 2003). Previous studies on schools used social statistical to explain the school population changes while this study used school data available from the ministry of education database. This helped to prove changes that occurred in school and thereafter predict the possible future changes that can occur depending on how certain parameter are handled.

5.2 Conclusion

This study considered a model of competition of students' population growth. The first and second category of population are the public schools' population and private schools' population respectively, which grows according to logistic growth equation. The third category is the non-enrolled entity which grows exponentially. In this work, eight equilibrium points were determined then studied for the local stability and global stability of the system. The Routh-Hurwitz criterion and eigenvalue method was use to evaluate the local stability of the equilibrium points. The local stability was verified by numerical simulation. Unstable equilibrium points were eight from the numerical simulation. However, the system was found to be globally asymptotically stable when solved using the Lyapunov function. From the analysis done the model was found to be in the positive region, the equilibrium points exist and globally asymptotically stable. The model was also bounded for all the variables. Thus this clearly depict the well-posedness of the model. Finally, from the results, this study suggest that government of Kenya through the ministry of education and any concerned body should consider implementing strategies that are aimed at reducing the parameters θ_2, θ_3 and η_3. This will save schools population from becoming to an extinct. This is important because education is the backbone of a country for the development purposes as the nation requires people who are wise and can help to solve the problems of the country. By employing strategies that mitigate these parameters, it will also help to prevent future occurrence of the same. Bifurcation was done for μ and which show coexistence when μ greater than 0.2.

In this work, we have we developed Lotka-Volterra model for the system of ODEs to asses to effects of predation, school dropout and competition among schools with Holling type II response. We used Matlab software to generate graphs. However, concerning our future studies, we intend to do real data fitting to the model. Another direction that is also of interest is to do optimal control theory and stochastic modelling.

5.3 Recommendation

Predation of schools' population especially from the non-enrolled category is a risky factor. Therefore, it is important to take action earlier enough to prevent schools' population from being overwhelmed by the situation and consequently coming to an extinct. This calls for proper dealing with factors that causes school predation and drop out. Competition between schools should be favorable.

The transfer of students to different schools should be avoided as it leads to time wastage and can cause school dropout. However, concerning our future studies, we intend to do real data fitting to the model. Another direction that is also of interest is to do optimal control theory and stochastic modelling.

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