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Decomposition of Riemannian Curvature Tensor Field and Its Properties

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Decomposition of recurrent curvature tensor fields of R-th order in Finsler manifolds has been studied by B. B. Sinha and G. Singh [1] in the publications del' institute mathematique, nouvelleserie, tome 33 (47), 1983 pg 217-220. Also Surendra Pratap Singh [2] in Kyungpook Math. J. volume 15, number 2 December, 1975 studied decomposition of recurrent curvature tensor fields in generalised Finsler spaces. Sinha and Singh [3] studied decomposition of recurrent curvature tensor fields in a Finsler space.

In this paper we study the Riemannian Curvature tensor with its properties its decomposition of the Riemannian curvature tensor and its properties. This raises important question: in Riemannian manifold V_n, is it possible to decompose Riemannian curvature tensor R^i_{jkl} of rank four, get another tensor of rank two and study its properties?

Keywords: Riemannian curvature tensor; decomposition.

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1 Introduction

In this study we investigate how to decompose the Riemannian curvature tensor using the decomposition tensor field $\varnothing_{jkl}$ and then study its properties $\varnothing_{jkl} = -\varnothing_{jlk}$, $\varnothing_{kl} = -\varnothing_{lk}$, $\nabla_n X^i = 0$, $\nabla_n \varnothing_{kl} + \nabla_k \varnothing_{nl} + \nabla_l \varnothing_{nk} = 0$. Next we use the above decomposition to study the different properties of the decomposition tensor i.e. it skew symmetric, recurrent and satisfies the Bianchi identity.

This Section begins with definition of spaces of N-dimension, coordinate transformations, tensor, covariant and contra- variant tensors, kronecker delta, symmetric and skew symmetric tensors, tensor contraction, a manifold, line element and metric tensor, conjugate tensor, christoffel symbols, tangent vector, tangent space, tensor field, affine connection, parallel transport and covariant differentiation.

Definition 1.1; Spaces of N-dimension

In a three dimensional space, a point is a set of three numbers called coordinates in a coordinate system or frame of reference. A coordinate system is a system which uses one or more numbers, or coordinates, to uniquely determine the position of the points or other geometric elements on a manifold such as Euclidean space. Similarly, a point in N-dimensional space is a set of N numbers noted by $(X^1, X^2, \ldots, X^N)$ [4].

Definition 1.2; Coordinate transformations

Let $(X^1, X^2, \ldots, X^N)$ and $((\bar{X}^1, \bar{X}^2, \bar{X}^3, \ldots, \bar{X}^N)$ be coordinates of a point in two different frames of reference. Suppose there exists N independent relations between them as under

$$\begin{aligned}\bar{X}^1 &= \bar{X}^1(X^1, X^2, \ldots, X^N)\\ \bar{X}^2 &= \bar{X}^2(X^1, X^2, \ldots, X^N)\\ \bar{X}^N &= \bar{X}^N(X^1, X^2, \ldots, X^N)\end{aligned}$$

which can be expressed as

$$\bar{X}^k = \bar{X}^k(X^1, X^2, \ldots, X^N),\ k = 1, 2, \ldots, N \tag{1.1}$$

where it is assumed that the functions are single – valued, continuous and have continuous derivatives.

Conversely to each set of coordinates

$(X^1, X^2, \ldots, X^N)$ there will corresponds a unique set of coordinates $(\bar{X}^1, \bar{X}^2, \bar{X}^3, \ldots, \bar{X}^N)$ given by

$$X^k = X^k(\bar{X}^1, \bar{X}^2, \bar{X}^3, \ldots, \bar{X}^N) \qquad [5] \tag{1.2}$$

Definition 1.3; Tensor

A tensor at a point on a coordinate manifold is a geometric object attached to that point satisfying certain properties. It is an abstract object having a definitely specified system of components in every coordinate system under considerations and such that under transformations of coordinates the components of the object undergo a transformation of certain nature. They describe relations between geometric objects [4].

Definition 1.4; Covariant and Contra-Variant Tensors

If N^2 quantities A^{ij} in a coordinate system $(X^1, X^2, \ldots, X^N)$ are related to other N^2 quantities $\bar{A}^{pq}$ in another coordinate system $(\bar{X}^1, \bar{X}^2, \ldots, \bar{X}^N)$ by the transformation equation

$$\bar{A}^{pq} = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial \bar{x}^q}{\partial x^j} A^{ij} \tag{1.3}$$

Then they are called contra-variant components of a tensor rank 2.

The N^2 quantities A_{ij} are called the components of a covariant tensor of rank two if they are related by the transformation equation;

$$A^{ij} = \frac{\partial x^i}{\partial \bar{x}^p}\frac{\partial x^j}{\partial \bar{x}^p}\bar{A}_{pq} \tag{1.4}$$

Similarly the N^2 quantities A_j^i are called the components of a mixed tensor of rank two (0ne contravariant and one covariant if

$$A_q^p = \frac{\partial \bar{x}^p}{\partial x^i}\frac{\partial x^i}{\partial x^q}A_j^i \quad [6] \tag{1.5}$$

Definition 1.5; The Kronecker Delta

The kronecker delta δ_j^i is a mixed tensor defined by

$$\delta_j^i = \begin{cases} o \; if \; i \neq j \\ 1 \; if \; i = j \end{cases} \tag{1.6}$$

Definition 1.6; Symmetric and skew – symmetric tensors

A tensor is said to be symmetric with respect to two contravariant or two covariant indices if its components remain unaltered upon interchange of indices. For example

$$A_{jk}^i = A_{kj}^i$$

A tensor is skew symmetric if

$$A_{jk}^i = -A_{kj}^i \quad [7]$$

Definition 1.7; Tensor Contraction

If one covariant and one contra-variant index of a mixed tensor are set equal, then the result indicates that the summation over the equal indices is to be taken, the resulting tensor is of rank two less than the original tensor. This process of reducing the rank of a tensor by two is called tensor contraction [4].

Definition 1.8; A Manifold

An n-dimensional Manifold, Vn (n-manifold) is a Housdorff – Topological connected space with the property that each point of it has a neighborhood which is homeomorphic to an Euclidean bowl En of n-dimensions [4].

Definition 1.9; The line-element and metric tensor

In a N – dimensional space, we define the line element as ds by the Quadratic form called the metric form as under:

$$ds^2 = \Sigma_{p=1}^{N}\Sigma_{q=1}^{N} g_{pq}dx^p dx^q \tag{1.7}$$

Or $ds^2 = g_{pq}dx^p dx^q$

here the quantity g_{pq} is called the components of covariant tensor of rank two known as the metric tensor [4].

Definition 1.10; Conjugate or Reciprocal metric tensor

Let g = $|g_{pq}|$ denote the determinant with elements g_{pq} and suppose g≠0. Then g^{pq} is defined as

$$g^{pq} = \frac{cofactor\ g_{pq}}{g} \tag{1.8}$$

g^{pq} is also asymmetric tensor known as conjugate tensor [4].

Definition 1.11; Christoffel symbols

The Christoffel symbols are tensor-like objects derived from a Riemannian metric g. They are used to study the geometry of the metric and appear, for example, in the geodesic equation. There are two closely related kinds of Christoffel symbols, the first kind Γ_{ijk}, and the second kind Γ^k_{ij}. Christoffel symbols of the second kind are also known as affine connections (Weinberg 1972, p. 71) or connection coefficients (Misner et al. 1973, p. 210).

These are numerical arrays of real numbers that describe, in coordinates the effects of parallel transport in curved surfaces and more generally, manifold represented by

$$\{k, i, j\} = ½\,(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \tag{i}$$

and

$$\left\{ \begin{matrix} l \\ ij \end{matrix} \right\} = \Gamma^l_{ij} = g^{lk}[k, ij] \tag{ii}$$

Equations (i) and (ii) are called the Christoffel symbols of the first and second kind respectively [8].

The christoffel symbols $[k, ij]$ and Γ^l_{ij} are symmetric in the indices j and k. The relation between is reciprocal. Through the following equations:

$$[j, ki] = g_{li}\Gamma^l_{ki}$$
$$g^{jm}[j, ki] = \Gamma^m_{ki}$$

Definition 1.12; Tangent vector

A tangent vector to a curve C passing through a point O of a smooth manifold at point O is ordered n–tuple of numbers $\left\{\frac{dx^i}{dt}\right\}_0$ $i = 1, 2, \ldots, N$

relative to the coordinate system $\{X^i\}$ at O.

Thus if $\hat{\lambda}$ is the tangent vector at O to C then

$$\hat{\lambda} = \left(\left(\frac{dx^1}{dt}\right)_0, \left(\frac{dx^2}{dt}\right)_0, \ldots, \left(\frac{dx^N}{dt}\right)_0\right) \qquad [9]$$

Definition 1.13; Tangent space

The set of all tangent vectors to the coordinate system $\{X^i\}$ to the curves of V_n passing through O forms an n-dimensional vector space T, called the tangent space at O [4].

Definition 1.14; Tensor field

If to each point of a region in N-dimensional space there correspond a definite tensor, then we say that a tensor field has been defined [4].

Definition 1.15; Affine connection

A geometric object on a smooth manifold which connects nearby tangent spaces and so permits tangent vectors fields to be differentiated as if they were functions on the manifolds [4].

Definition 1.16; Parallel transport

A way of transporting geometrical data along smooth curves in a manifold [4].

Definition 1.17; Covariant differentiation

This is a way of introducing and working with a connection on a manifold by means of a differential operator. The covariant derivatives represent the rates of change of physical quantities independent of any frames of reference. The covariant derivative of the tensor A^p and A_q are defined as

$$\nabla_q A^p = \partial_q A^p + \Gamma^p_{qs} A^s$$

$$\nabla_r A_q = \partial_r A_q - \Gamma^s_{qs} A_s$$

Similarly covariant differentiation of a mixed tensor$A^p_q$is given by

$$\nabla_r A^p_q = \partial_r A^p_q + \Gamma^p_{rs} A^s_{qr} - \Gamma^s_{qr} A^p_s \qquad [10]$$

2 Methodology

This Section begins with definition of Riemannian manifold, Riemannian curvature tensor …

2.1 Definition of Riemannian manifold

In differential geometry a Riemannian manifold or Riemannian space $(M, g)$is a real smooth manifold M equipped with an inner product g_p on the tangent space$T_p M$ at each point P that varies smoothly from point to point in the sense, that is if X and Y are vector fields on M, then $P \to g_p\big(X(P), Y(P)\big)$is a smooth function [9]. The family g_p of inner products is called a Riemannian metric. It can also be defined as a smooth manifold with a smooth section of the positive definite quadratic forms on the tangent bundle [4].

2.2 Definition of Riemannian curvature tensor

Curvature in mathematics refers intuitively to the amount by which a geometric object deviates from being flat, or straight in the case of a line. In a plane this is a scalar quantity, but in three or more dimensions it is described by a curvature vector that takes into account the direction of the bend as well as its sharpness [4].

The curvature of more complex objects (such as curved n-dimensional spaces such as Riemannian manifolds) is described by more complex objects. For Riemannian manifolds the curvature is described by Riemannian curvature tensor.

Riemannian curvature tensor associates a tensor at each point of a Riemannian manifold that measures the extent to which the metric tensor is not locally isometric to a Euclidian space. It is a central mathematical tool in the theory of the General Relativity, the Modern Theory of Gravity and the curvature of space time is in principle observable via the geodesic deviation equation. The curvature tensor represents the tidal force experienced by a rigid body moving along a geodesic [4].

The curvature tensor with respect to Christoffel symbols has components

$$R^i_{jkl} \text{given by} R^i_{jkl} = \partial_j \Gamma^i_{kl} - \partial_k \Gamma^i_{jl} + \Gamma^i_{pj}\Gamma^p_{kl} - \Gamma^i_{pk}\Gamma^p_{jl} \tag{2.1.1}$$

The tensor is called Riemannian curvature tensor or Riemannian – Christoffel tensor of the second kind [4].

2.3 Properties of Riemannian curvature tensor

The Riemannian curvature tensor satisfies the following identities

$$R^i_{jkl} = -R^i_{kjl} \tag{2.2.1}$$

$$R^i_{jkl} + R^i_{ljk} + R^i_{klj} = 0 \tag{2.2.2}$$

$$\nabla_S R^i_{jkl} + \nabla_k R^i_{sjl} + \nabla_j R^i_{ksl} = 0 \tag{2.2.3}$$

Equations (2.2.2) and (2.2.3) are called Bianchi's first and second identities [4].

The covariant derivative of the Riemannian curvature tensor $R^i_{jkl}$is defined as

$$\nabla_m R^i_{jkl} = \partial_m R^i_{jkl} + R^s_{jkl}\Gamma^i_{ms} - R^i_{skl}\Gamma^s_{mj} - R^i_{jsl}\Gamma^s_{mk} - R^i_{jks}\Gamma^s_{ml} \qquad [4] \tag{2.2.4}$$

The commutation laws involving the curvature tensor field R^i_{jkl} are given by

$$\nabla_j \nabla_k \lambda^i - \nabla_k \nabla_j \lambda^i = \lambda^l R^i_{jkl} \tag{2.2.5}$$

$$\nabla_j \nabla_k \lambda^i_t - \nabla_k \nabla_j \lambda^i_t = \lambda^l_t R^i_{jkl} - \lambda^i_l R^l_{jkt} \tag{2.2.6}$$

$$2\nabla_{[j)\nabla_{(k]}} \lambda^i = \lambda^i R^i_{jkl} \tag{2.2.7}$$

$$2\nabla_{[j)\nabla_{(k]}} \emptyset_i = - \emptyset_i R^i_{jkl} \tag{2.2.8}$$

The equations (2.2.7) and (2.2.8) are known as Ricci laws for covariant differentiation .Lambda is the component of any vector tangential to the surface [4].

The Riemannian curvature tensor is also skew symmetric with respect to its first two and last two indices that is

$$R^i_{jkl} = -R^i_{kjl} \tag{2.2.9}$$

$$R^i_{jkl} = -R^i_{jlk} \tag{2.2.10}$$

The Riemannian curvature tensor in which there exists a non zero vector V_n such that the curvature tensor satisfy the relation.

$$\nabla_n R^i_{jkl} = \mathrm{V}_n R^i_{jkl} \tag{2.2.11}$$

is said to be recurrent and the curvature tensor field of the space called recurrent tensor field.

R^i_{jkl} satisfies the following theorems.

Theorem 2.1

If the associated curvature has components

$$R_{jklm} = R^n_{jkl} g_{nm} \tag{2.2.12}$$

Then R_{jklm} is

Skew symmetric in the first two indices

$$R_{(jk)lm} = 0 \tag{2.2.13}$$

Skew symmetric in the last two indices

$$R_{jk(lm)} = 0 \tag{2.2.14}$$

Satisfy Bianchi's identities

$$R_{[jkl]m} = 0 \tag{2.2.15}$$

$$\nabla_{[p} R_{jk]lm} = 0 \tag{2.2.16}$$

Symmetric in two parts of indices

$$R_{jklm} = R_{lmjk} \tag{2.2.17}$$

Proof

Using (2.1.1) and (2.2.2) we have

$$R_{jklm} = 2g_{mn}\, \partial_{[j}\, \Gamma^n_{k]l} + 2g_{mn}\, \Gamma^m_{p[j}\, \Gamma^p_{k]l}$$

$\partial_{[j}\, \Gamma^n_{k]l}$ is skew symmetric with respect to indices j and k. and thus in expanded for it can be expressed as

$$R_{jklm} = 2g_{mn} \times \frac{1}{2}\left[\, \partial_j \Gamma^n_{kl} - \partial_k \Gamma^n_{jl}\,\right] + 2g_{mn} \times \frac{1}{2}\left[\, \Gamma^m_{pj} \Gamma^p_{kl} - \Gamma^m_{pk} \Gamma^p_{jl}\,\right]$$

Alternatively we obtain

$$\begin{aligned}
&R_{jklm} = g_{mn}\, \partial_j \Gamma^n_{kl} - g_{mn}\, \partial_k \Gamma^n_{jl} + g_{mn}\, \Gamma^n_{pj} \Gamma^p_{kl} - g_{mn}\, \Gamma^n_{pk} \Gamma^p_{jl} \\
&R_{jklm} + R_{kjlm} = g_{mn}\, \partial_j \Gamma^n_{kl} - g_{mn}\, \partial_k \Gamma^n_{jl} + g_{mn}\, \Gamma^n_{pj} \Gamma^p_{kl} - g_{mn}\, \Gamma^n_{pk} \Gamma^p_{jl} \\
&+ g_{mn}\, \partial_k \Gamma^n_{jl} - g_{mn}\, \partial_j \Gamma^n_{kl} + g_{mn}\, \Gamma^n_{pk} \Gamma^p_{jl} - g_{mn}\, \Gamma^n_{pj} \Gamma^p_{kl} = 0 \\
&R_{jklm} + R_{kjlm} = 0
\end{aligned}$$

This is equivalent to

$$R_{jk(lm)} = 0$$

ii) In view of RICCI identities we find

$$2\,\nabla_{[[\ j}\nabla_{k\ ]]}\, g_{mn} = -g_{mn}\ R^{p}_{jkm} - g_{mn}\, R^{p}_{jkn}$$

Using $\nabla_k\, g_{mn} = 0$ and equation (2.2.2) we obtain

$$0 = -R_{jkmn} + R_{jknm}$$

Or equivalently in symmetric bracket

$$0 = R_{jk(mn)}$$

which is same as (2.2.14)

iii) If we multiply equation (2.2.2) and (2.2.3) by g_{mn} and sum with respect to m we obtain (2.2.6) and (2.2.7) respectively,

iv) The equation (2.2.15) is equivalent to

$$R_{jklm} + R_{kljm} + R_{ljkm} = 0$$

The three similar equations are expressed in the form

$$R_{jkml} + R_{lmkj} + R_{mklj} = 0$$
$$R_{lmjk} + R_{mjlk} + R_{jlmk} = 0$$
$$R_{mjkl} + R_{jkml} + R_{kmjl} = 0$$

Adding above results we find

$$R_{jklm} + R_{kljm} + R_{ljkm} + R_{jklm} + R_{kljm} + R_{ljkm} + R_{lmjk} + R_{mjlk} + R_{jlmk} + R_{mjkl} + R_{jkml} + R_{kmjl} = 0$$

Using (2.2.3) and (2.2.14)

Or

$$R_{ljkm} - R_{kmlj} - R_{jlkm} - R_{kmlj} = 0$$

Or

$$R_{ljkm} + R_{ljkm} - R_{kmlj} - R_{kmlj} = 0$$

Or

$$2R_{ljkm} - 2R_{kmlj} = 0$$

Which proves $R_{ljkm} = R_{kmlj}$

Which is the same as (2.2.17)

Before stating the second theorem we prove that the Christofell symbol

$$\Gamma^{i}_{jk} = ½\, \partial_{\mathrm{j}} \log\ \mathrm{g} = \partial_{\mathrm{j}}\ \log\sqrt{g} \ \text{ where }\ g = |g_{ij}|$$

Proof

From the definition 1.1.7 we know that

$$g^{jk} = \frac{\text{co-factor of } g_{jk}}{g}$$

Or

$$g^{jk} = \frac{G_{(jk)}}{g}$$

Or $g\, g^{jk} = G_{(j,k)}$…….(*)

Multiplying the above equation (*) with g_{jr} we obtain

$$g\, g^{jk}\, g_{jr} = G_{(j,k)} g_{jr}$$

Or

$$g\, \partial_r^k = G_{(j,k)} g_{jr} \text{ (for k = r) we have}$$

$$g = G_{(j,r)} g_{jr}$$

whose differentiation with respect to X^m gives

$$\frac{\partial g}{\partial x^m} = G_{(j,k)} \frac{\partial g_{jr}}{\partial x^m}$$

The above result can be expanded as

$$\frac{\partial g}{\partial x^m} = g\; g^{jr}\, ([\, j, rm] + [\, r, jm])$$

Considering the effect of conjugate metric tensor we find,

$$\frac{\partial g}{\partial x^m} = g\left\{ {r \atop rm} \right\} + g\left\{ {r \atop jm} \right\}$$

The above result is equivalent to

$$\partial_m g = 2\, g\, \Gamma_{rm}^r \qquad \left(\text{Since } g\left\{ {r \atop rm} \right\} = \Gamma_{rm}^r\right)$$

Which is same as

$$1/_2\, g\, \partial_m\, g = \Gamma_{rm}^r$$

Or

$$\partial_m\, Log\, \sqrt{g} = \Gamma_{rm}^r$$

Theorem 2.2

Riemannian curvature tensor of second kind can be contracted in two ways, that is, one yielding a zero and other a systematic tensor.

Proof

Contracting indices n and l, equation (2.1.1) we find

$$C^l{}_3\, R_{jkl}^n = C^l{}_3\, [\, 2\partial_{[j}\, \Gamma_{k\,]l}^n + 2\Gamma_{p\,[j}^n\, \Gamma_{k\,]l}^p$$

Or

$$R^{l}_{jkl} = 2\partial_{[j}\,\Gamma^{l}_{k\;]l} + 2\Gamma^{n}_{p\,[j}\,\Gamma^{p}_{k\;]l} \tag{2.2.18}$$

We know that

$$\Gamma^{l}_{kl} = {}^{1}\!/_{2}\,\partial_{k}\,\log g$$

$$\partial_{[j}\,\Gamma^{l}_{k\;]l} = {}^{1}\!/_{2}\,\partial_{[j}\partial_{k]}\quad \log g = 0 \tag{2.2.19}$$

Also

$$2\Gamma^{l}_{p\,[j}\,\Gamma^{p}_{k\;]l} = 2\Gamma^{p}_{l\,[j}\,\Gamma^{l}_{k\;]p}$$

Or

$$2\times{}^{1}\!/_{2}\left[\Gamma^{l}_{pj}\Gamma^{p}_{kl} - \Gamma^{l}_{pk}\Gamma^{p}_{jl}\right] = 2\times{}^{1}\!/_{2}\left[\Gamma^{p}_{lj}\Gamma^{l}_{kp} - \Gamma^{p}_{lk}\Gamma^{l}_{jp}\right] = 0 \tag{2.2.20}$$

Using (2.2.2.) and (2.2.3) in (2.2.1) gives

$$R^{l}_{jkl} = 0 \tag{2.2.21}$$

which proves the first part

The equation (2.1.4) which Bianchi identify is equivalent to

$$R^{m}_{jkl} + R^{m}_{klj} + R^{m}_{ljk} = 0$$

Setting m = j in the above equation gives

$$R^{j}_{jkl} + R^{j}_{klj} + R^{j}_{ljk} = 0 \tag{2.2.22}$$

But $R^{j}_{klj} = 0$ in view of (2.2.2), thus above equation reduces to $R^{j}_{jkl} + R^{j}_{ljk} = 0$

In view of skew symmetry property of R^{j}_{jkl} we get

$$R^{j}_{jkl} - R^{j}_{jlk} = 0$$

Contracting the above equation it becomes

$$R_{kl} + R_{lk} = 0$$

$$R^{j}_{[kl]} = 0$$

Where [ ] is skew symmetric bracket which proved the last part.

3 Decomposition of Riemannian Curvature Tensor R^{i}_{jkl}

Decomposition is a way of breaking up of the Riemannian curvature tensor into pieces with useful individual algebraic properties. It is the decomposition of the space of all tensors having the symmetries of the Riemannian tensor into its irreducible representation for the action of the orthogonal group.

Let's consider the decomposition of the Riemannian curvature tensor R^i_{jkl} in the following form

$$R^i_{jkl} = X^i \emptyset_{jkl} , \tag{3.1.1}$$

Where $\emptyset_{jkl}$ is the decomposition tensor field and X^i is a vector field such that

$$X^i V_i = 1 \tag{3.1.2}$$

According to the Kronecker delta definition (1.1.5)

3.1 Properties of decomposition tensor field $\emptyset_{jkl}$

If we multiply (3.1.1.) with V_i we will obtain

$$V_i R^i_{jkl} = V_i X^i \emptyset_{jkl}$$

Since $X^i V_i = 1$ the above equation becomes

$$V_i R^i_{jkl} = \emptyset_{jkl} \tag{3.1.3}$$

By interchanging the indices k and l and adding in the above equation we get

$$V_i R^i_{jkl} + V_i R^i_{jlk} = \emptyset_{jkl} + \emptyset_{jlk}$$

Or

$$V_i (R^i_{jkl} + R^i_{jlk}) = \emptyset_{jkl} + \emptyset_{jlk} \tag{3.1.4}$$

Since R^i_{jkl} is skew symmetric in the last two indices, that is,

$R^i_{jlk} = -R^i_{jkl}$ in view of (2.2.10)

Using the equation (2.2.10) in the equation (3.1.4) we have

$$V_i (R^i_{jkl} - R^i_{jkl}) = \emptyset_{jkl} + \emptyset_{jlk}$$

which reduces to

$0 = \emptyset_{jkl} + \emptyset_{jlk}$

This yields the following identity

$$\emptyset_{jkl} = -\emptyset_{jlk} \tag{3.1.5}$$

Thus we state the following theorems

Theorem 3.1.1

In a Riemannian space the decomposition tensor is skew symmetric in the last two indices, that is

$\emptyset_{jkl} = -\emptyset_{jlk}$

We further decompose the tensor field $\emptyset_{jkl}$ as under

$$\emptyset_{jkl} = V_j \emptyset_{kl} \tag{3.1.6}$$

Multiplying (3.1.6) by X^j we obtain

$$X^j \emptyset_{jkl} = X^j V_j \emptyset_{kl}$$

In view of (3.1.2) the above equation yields

$$X^j \emptyset_{jkl} = \emptyset_{kl} \tag{3.1.7}$$

By interchanging the two indices k and l in (3.1.7) and adding the respective results we get

$$X^j (\emptyset_{jkl} + \emptyset_{jlk}) = \emptyset_{kl} + \emptyset_{lk}$$

in view of (3.1.5) we have

$$X^j (\emptyset_{jkl} - \emptyset\_jlk) = \emptyset_{kl} + \emptyset_{lk} ,$$

which reduces

$$0 = \emptyset_{kl} + \emptyset_{lk} ,$$

This implies that

$$\emptyset_{kl} = - \emptyset_{lk} \tag{3.1.8}$$

We state

Theorem 3.1.2

The decomposition tensor $\emptyset_{kl}$ is skew symmetric with respect to its two indices k and l, that is

$$\emptyset_{kl} = - \emptyset_{lk}$$

Using equation (2.2.2) and (3.1.1) we get the following equation

$$X^i(\emptyset_{jkl} + \emptyset_{klj} + \emptyset_{ljk}) = 0 \tag{3.1.9}$$

Transvecting (3.1.9) by V_i we get

$$X^i V_i (\emptyset_{jkl} + \emptyset_{klj} + \emptyset_{ljk}) = 0 , \tag{3.1.10}$$

which becomes

$$\emptyset_{jkl} + \emptyset_{klj} + \emptyset_{ljk} = 0$$

In view of (3.1.2)

Thus we have

Theorem 3.1.3

The decomposition tensor $\emptyset_{jkl}$ satisfies the identity

$$\emptyset_{jkl} + \emptyset_{klj} + \emptyset_{ljk} = 0$$

Taking covariant differentiation of (3.1.1) with respect to X^n and making use of (2.2.11) we obtain

$$\nabla_{\mathrm{n}} R^i_{jkl} = \nabla_{\mathrm{n}} X^i \emptyset_{jkl} + X^i \nabla_{\mathrm{n}} \emptyset_{jkl} \tag{3.1.12}$$

Consider X^i to be covariant constant and using (3.1.1) the equation (3.1.12) gives

$$\nabla_{\mathrm{n}} R^i_{jkl} = \nabla_{\mathrm{n}} \emptyset_{jkl} \tag{3.1.13}$$

By virtue of (3.1.1) the equation (3.1.12) yield

$$\emptyset_{jkl} \nabla_{\mathrm{n}} X^i = 0 \tag{3.1.14}$$

Since $\emptyset_{jkl} \neq 0$ therefore we have

$$\nabla_{\mathrm{n}} X^i = 0 \tag{3.1.15}$$

That is X^i is covariant constant

Hence we conclude

Theorem 3.1.4

The necessary and sufficient condition for the decomposition tensor field $\emptyset_{jkl}$ and $\emptyset_{kl}$ to be recurrent is that the vector field X^i is covariant.

In view of (3.1.1), (2.1.12) and

$$R^i_{jkl} = X^i \emptyset_{jkl}$$

The Bianchi identity of the form

$\nabla_{\mathrm{n}} R^i_{jkl} + \nabla_{\mathrm{K}} R^i_{jnl} + \nabla_{\mathrm{n}} R^i_{jnk} = 0$ becomes

$$X^i \left[\nabla_{\mathrm{n}} \emptyset_{jkl} + \nabla_{\mathrm{k}} \emptyset_{jnl} + \nabla_{\mathrm{l}} \emptyset_{jnk}\right] = 0 \tag{3.1.16}$$

Transvecting (3.1.16) by h^j it gives

$$X^i \left[\nabla_{\mathrm{n}} \emptyset_{kl} + \nabla_{\mathrm{k}} \emptyset_{nl} + \nabla_{\mathrm{l}} \emptyset_{nk}\right] = 0 \tag{3.1.17}$$

Now under the assumption that X^i is covariant constant, the equation (3.1.17) reduces to

$$\nabla_{\mathrm{n}} \emptyset_{kl} + \nabla_{\mathrm{k}} \emptyset_{nl} + \nabla_{\mathrm{l}} \emptyset_{nk} = 0 \tag{3.1.18}$$

Which is the Bianchi identity for the Decomposition tensor field of $\nabla_{\mathrm{n}} \emptyset_{kl}$. Thus we have

Theorem 3.1.5

The decomposition tensor field satisfies the Bianchi identity

$$\nabla_n \emptyset_{kl} + \nabla_k \emptyset_{nl} + \nabla_l \emptyset_{nk} = 0$$

4 Application, Conclusion and Suggestions for Further Research

4.1 Application

Like most mathematicians, Riemannian Geometers look for theorems even when there are no practical applications. The theorems like the fundamental theorem of algebra that can be used to study gravitational lensing are much older than Einstein's equation and the Hubble telescope.

Einstein, for example, studied Riemannian geometry before he developed his theories. His equations involve a special curvature called Ricci curvature, which was first defined by mathematicians.

We expect that practical applications of our theorems will be discovered some day in the future. Without having mathematical theorems sitting around for them to apply, Physicist would have trouble discovering new theories and describing them.

4.2 Summary and conclusion

In chapter two to this study, we have defined curvature tensor in Riemannian space with respect to Christoffel symbols. In the same chapter we have highlighted identities satisfied by the Riemannian curvature tensor which includes; skew symmetric property and the Bianchi identities we have also stated the commutative laws involving the curvature tensor field R^i_{jkl} in chapter three we have defined the decomposition tensor field. We have then applied commutation laws, Bianchi identities had skew symmetric identities to the decomposition tensor field $\emptyset_{jkl}$ and found that:

$R^i_{jkl} = X^i \emptyset_{jkl}$ (Decomposition definition)

$\emptyset_{jkl} = -\emptyset_{jlk}$ (Skew symmetric property of $\emptyset_{jkl}$)

$\emptyset_{kl} = -\emptyset_{lk}$ (Skew symmetric property of $\emptyset_{kl}$)

$\emptyset_{jkl} + \emptyset_{klj} + \emptyset_{ljk} = 0$ (Identity similar to first Bianchi identity)

$\nabla_n \emptyset_{kl} + \nabla_k \emptyset_{nl} + \nabla_l \emptyset_{nk} = 0$ (Identity similar to second Bianchi identity)

4.3 Suggestion for area of further research

Having investigated some properties of decomposition tensor field in Riemannian spaces we can extend this work by investigating properties of decomposition tensor field in recurrent Riemannian spaces and also decomposition of Berward and Weyl curvature tensor fields.

Competing Interests

Authors have declared that no competing interests exist.

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