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Tensor Product of 2-Frames in 2-Hilbert Spaces

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Abstract

2-frames in 2-Hilbert spaces are studied and some results on it are presented. The tensor product of 2-frames in 2-Hilbert spaces is introduced. It is shown that the tensor product of two 2-frames is a 2-frame for the tensor product of Hilbert spaces. Some results on tensor product of 2-frames are established.

Keywords

Tensor Product, 2-Inner Product Spaces, Frames, 2-Frames

1. Introduction

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaefer in 1952 to study some deep problems in nonharmonic Fourier series. D. Han and D.R. Larson [1] have developed a number of basic aspects of operator-theoretic approach to frame theory in Hilbert space. Peter G. Casazza [2] presented a tutorial on frame theory and he suggested the major directions of research in frame theory.

The concept of linear 2-normed spaces has been investigated by S. Gahler in 1965 [3] and has been developed extensively in different subjects by many authors. A concept which is related to a 2-normed space is 2-inner product space which has been intensively studied by many mathematicians in the last three decades. The concept of 2-frames for 2-inner product spaces was introduced by Ali Akbar Arefijammaal and Ghadir Sadeghi [4] and described some fundamental properties of them. Y. J. Cho, S. S. Dragomir, A. White and S. S. Kim [5] are presented some inequalities in 2-inner product spaces. Some results on 2-inner product spaces are described by H. Mazaherl and R. Kazemi [6]. The tensor product of frames in tensor product of Hilbert spaces is introduced by G. Upender Reddy and N. Gopal Reddy [7] and some results on tensor frame operator are presented.

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In this paper, 2-frames in 2-Hilbert spaces are studied and some results on it are presented. The tensor product of 2-frames in 2-Hilbert spaces is introduced. It is shown that the tensor product of two 2-frames is a 2-frame for the tensor product of Hilbert spaces. Some results on tensor product of 2-frames are established.

2. Preliminaries

The following definitions from [2] [5] are useful in our discussion.

Definition 2.1. A sequence $\{x_i\}_{i=1}^{\infty}$ of vectors in a Hilbert space X is called a frame if there exist constants $0 < A \le B < \infty$ such that

$$A\|x\|^2 \le \sum_{i=1}^{\infty} \left|\langle x, x_i \rangle\right|^2 \le B\|x\|^2 \quad \text{for all } x \in X.$$

The above inequality is called the frame inequality. The numbers A and B are called lower and upper frame bounds respectively.

Definition 2.2. A synthesis operator $T: l_2 \to X$ is defined as $Te_i = x_i$ where $\{e_i\}$ is an orthonormal basis for l_2.

Definition 2.3. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for X and $\{e_i\}$ be an orthonormal basis for l_2. Then, the analysis operator $T^*: X \to l_2$ is the adjoint of synthesis operator T and is defined as $T^*x = \sum_{i=1}^{\infty} \langle x, x_i \rangle e_i$ for all $x \in X$.

Definition 2.4. Let $\{x_i\}_{i=1}^{\infty}$ be a frame for the Hilbert space H. A frame operator $S = TT^* : X \to X$ is defined as $Sx = \sum_{i=1}^{\infty} \langle x, x_i \rangle x_i$ for all $x \in X$.

Here we give the basic definitions of 2-normed spaces and 2-inner product spaces from [3] [6].

Definition 2.5. X be a real linear space of dimension greater than 1 and let $\|.,.\|$ be a real-valued function on $X \times X$ satisfying the following conditions.

a) $\|x, y\| \ge 0$ and $\|x, y\| = 0$ if and only if x and y are linearly dependent vectors.

b) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

c) $\|\alpha x, y\| = |\alpha| \|x, y\|$ for any real number α and for all $x, y \in X$.

d) $\|x + y, z\| \le \|x, z\| + \|y, z\|$ for all $x, y, z \in X$.

Then $\|.,.\|$ is called 2-norm on X and $(X, \|.,.\|)$ called a linear 2-normed space.

Definition 2.6. Let X be a linear space of dimension greater than 1 over the field K (=R or C). Suppose that $(.,./.)$ is K-valued function on $X \times X \times X$ which satisfies the following conditions.

a) $(x, x/z) \ge 0$ and $(x, x/z) = 0$ if and only if x and z are linearly dependent.

b) $(x, x/z) = (z, z/x)$.

c) $(y, x/z) = \overline{(x, y/z)}$.

d) $(\alpha x, y/z) = \alpha (x, y/z)$ for all $\alpha \in K$.

e) $(x_1 + x_2, y/z) = (x_1, y/z) + (x_2, y/z)$.

Then $(.,./.)$ is called a 2-inner product on X and $(X, (.,./.))$ is called a 2-inner product space (or 2-pre Hilbert space).

If $(X, \langle\ \rangle)$ is an inner product space, then the standard 2-inner product space $(.,./.)$ is defined on X by

$$(x, y/z) = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle \text{ for all } x, y, z \in X .$$

Let $(X, (.,./.))$ be a 2-inner product space, we can define a 2-norm on $X \times X$ by $\|x, y\| = (x, x/y)^{\frac{1}{2}}$, for all $x, y \in X$.

Using the above properties, we can prove the Cauchy-Schwartz inequality $(x, y/b)^2 \le \|x, b\|^2 \|y, b\|^2$.

A 2-inner product space X is called a 2-Hilbert space if it is complete.

3. 2-Frames

The definition of 2-frame from [1] as follows.

Definition 3.1 Let $(X,(.,./.))$ be a 2-Hilbert space and $\xi \in X$. A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in X is called a 2-frame associated to ξ if there exist $0 < A \le B < \infty$ such that

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, x_i/\xi)|^2 \le B\|x,\xi\|^2 \text{ for all } x \in X.$$

The above inequality is called the 2-frame inequality. The numbers A and B are called the lower and upper 2-frame bounds respectively.

The following proposition [1] shows that in the standard 2-inner product spaces every frame is a 2-frame.

Proposition 3.2. Let $(X,\langle\ \rangle)$ be a Hilbert space and $\{x_i\}_{i=1}^{\infty}$ is a frame for *H*. Then, it is a 2-frame with the standard 2-inner product space on *X*.

Proof: Suppose that $\{x_i\}_{i=1}^{\infty}$ is a frame for *X* with frame bounds *A* and *B*.

$$\sum_{i=1}^{\infty} |(x, x_i/\xi)|^2 = \sum_{i=1}^{\infty} |\langle x, x_i\rangle\langle \xi,\xi\rangle - \langle x,\xi\rangle\langle \xi, x_i\rangle|^2 = \sum_{i=1}^{\infty} |\langle x, x_i\rangle - \langle x,\xi\rangle\langle \xi, x_i\rangle|^2$$

Then
$$= \sum_{i=1}^{\infty} |\langle x - \langle x,\xi\rangle \xi, x_i\rangle|^2 \le B\|x - \langle x,\xi\rangle\xi\|^2 = B\left(\|x\|^2 - |\langle x,\xi\rangle|^2\right)$$

$$= B(x, x/\xi) = B\|x,\xi\|^2.$$

Similarly we can prove that $A\|x,\xi\|^2 \le \sum_{i=1}^{\infty} |(x, x_i/\xi)|^2$. Hence $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for 2-Hilbert space. □

Suppose $(X,(.,./.))$ is a 2-Hilbert space and L_ξ the subspace generated with a fixed element ξ in *X*. Let M_ξ be denote the algebraic complement of L_ξ in *X*. So we have $L_\xi \oplus M_\xi = X$ .We define the inner product $\langle .,.\rangle_\xi$ on *X* as follows $\langle x, z\rangle_\xi = \langle x, z/\xi\rangle$.

A sequence $\{x_i\}_{i=1}^{\infty}$ of elements in *X* is a 2-frame associated to ξ with frame bounds *A* and *B*, then the definition of 2-frame can be written as $A\|x\|_\xi^2 \le \sum_{i=1}^{\infty} |(x, x_i)_\xi|^2 \le B\|x\|_\xi^2$, for all $x \in X$.

Definition 3.3. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in *X*. Then, the 2-Synthesis operator $T_\xi : l^2 \to X_\xi$ is defined by $T_\xi\{c_i\} = \sum_{i=1}^{\infty} c_i x_i$.

Definition 3.4. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame in *X*. Then, the 2-Analysis operator $T_\xi^* : X_\xi \to l^2$ is defined by $T_\xi^*(x) = \{(x, x_i/\xi)\}_{i=1}^{\infty}$.

Definition 3.5. Let $\{x_i\}_{i=1}^{\infty}$ be a 2-frame associated to ξ with frame bounds *A* and *B* in a 2-Hilbert space *X*. A 2-frame operator $S_\xi : X_\xi \to X_\xi$ is defined by $S_\xi x = \sum_{i=1}^{\infty} (x, x_i/\xi) x_i$.

Theorem 3.6. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a sequence in 2-Hilbert space *X*, with $x = \sum_{i=1}^{\infty} (x, x_i/\xi) x_i$ holds for all $x \in X$ if and only if $\{x_i\}_{i=1}^{\infty}$ is a 2-normalized tight frame for *X*.

Proof: Since $\{x_i\}_{i=1}^{\infty}$ is a 2-normalized tight frame for *X*, for all $x \in X$

$$\Leftrightarrow \|x,\xi\|^2 = \sum_{i=1}^{\infty} |(x, x_i/\xi)|^2 \Leftrightarrow \|x,\xi\|^2 = \sum_{i=1}^{\infty} (x, x_i/\xi)(x_i, x/\xi)$$

$$\Leftrightarrow (x, x/\xi) = \left(\sum_{i=1}^{\infty} (x, x_i/\xi) x_i, x/\xi\right) \Leftrightarrow x = \sum_{i=1}^{\infty} (x, x_i/\xi) x_i \text{ for all } x \in X. \ \square$$

Theorem 3.7. Suppose that $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for Hilbert space *X*, and *T* is co-isometry. Then $\{Tx_i\}_{i=1}^{\infty}$ is a 2-frame for *X*.

Proof: Since $\{x_i\}_{i=1}^{\infty}$ is a 2-frame for *X*, by Definition 3.1, we have

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty}\left|\left(x,x_i/\xi\right)\right|^2 \le B\|x,\xi\|^2, \left(x\in X\right) \tag{1}$$

Since $T^*: X \to X$ is an operator, for all $x \in H$, we have $T^*x \in X$

Therefore, the above Equation (1) is true for $T^*x \in X$

$$A\left\|T^*x,\xi\right\|^2 \le \sum_{i=1}^{\infty}\left|\left(T^*x,x_i/\xi\right)\right|^2 \le B\left\|T^*x,\xi\right\|^2$$

$$A\left\|T^*x,\xi\right\|^2 \le \sum_{i=1}^{\infty}\left|\left(x,Tx_i/\xi\right)\right|^2 \le B\left\|T^*x,\xi\right\|^2, \text{ for all } x\in X$$

By using the fact that T is co-isometry, we have

$$A\|x,\xi\|^2 \le \sum_{i=1}^{\infty}\left|\left(x,Tx_i/\xi\right)\right|^2 \le B\|x,\xi\|^2, \text{for all } x\in X$$

Which shows that $\{Tx_i\}_{i=1}^{\infty}$ is a 2-frame for *X*. □

4. Tensor Product of 2-Frames

Let H_1 and H_2 be 2-Hilbert spaces with inner products $(.,./.)_1$, $(.,./.)_2$ respectively. The tensor product of H_1 and H_2 is denoted by $H_1 \otimes H_2$ and is an inner product space with respect to the inner product given by

$$\left(x_1\otimes x_2, y_1\otimes y_2/z_1\otimes z_2\right) = \left(x_1,y_1/z_1\right)_1\left(x_2,y_2/z_2\right)_2 \tag{2}$$

for all $x_1,y_1,z_1 \in H_1$ and $x_2,y_2,z_2 \in H_2$. The norm on $H_1 \otimes H_2$ is defined by

$$\|x_1\otimes x_2, y_1\otimes y_2\| = \|x_1,y_1\|_1\|x_2,y_2\|_2 \quad \forall x_1,y_1\in H_1 \text{ and } x_2,y_2\in H_2 \tag{3}$$

where $\|.,.\|_1$, and $\|.,.\|_2$ are norms generated by $(.,./.)_1$ and $(.,./.)_2$ respectively. The space $H_1 \otimes H_2$ is completion with the above inner product. Therefore, the space $H_1 \otimes H_2$ is a 2-Hilbert space.

The following definition is the extension of (3.1) to the sequence $\{x_i \otimes y_j\}$.

Definition 4.1. Let $\{x_i\}$ and $\{y_j\}$ be the sequences of vectors in 2-Hilbert spaces H_1 and H_2 respectively. Then, the sequence of vectors $\{x_i \otimes y_j\}$ is said to be a tensor product of 2-frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ associated to $\xi \otimes \eta$ if there exist two constants $0 < A \le B < \infty$ such that

$$A\|x\otimes y,\xi\otimes\eta\|^2 \le \sum_{i,j}\left|\left(x\otimes y, x_i\otimes y_j/\xi\otimes\eta\right)\right|^2 \le B\|x\otimes y,\xi\otimes\eta\|^2,$$
$$\text{for all } x\otimes y \in H_1\otimes H_2$$

The numbers *A* and *B* are called lower and upper frame bounds of the tensor product of 2-frame, respectively.

Theorem 4.2. Let $\{x_i\}$ and $\{y_j\}$ be two sequences in Hilbert spaces H_1 and H_2 respectively. Then, the sequence $\{x_i \otimes y_j\}$ is a tensor product of 2-frame for $H_1 \otimes H_2$ if and only if $\{x_i\}$ and $\{y_j\}$ are the 2-frames for H_1 and H_2 respectively.

Proof. Suppose that $\{x_i \otimes y_j\}$ is a 2-frame for $H_1 \otimes H_2$ associated to $\xi \otimes \eta$. Then, for each $x \otimes y \in H_1 \otimes H_2 - \{0 \otimes 0\}$

$$A\|x\otimes y,\xi\otimes\eta\|^2 \le \sum_{i,j}\left|\left(x\otimes y, x_i\otimes y_j/\xi\otimes\eta\right)\right|^2 \le B\|x\otimes y,\xi\otimes\eta\|^2,$$
$$\text{for all } x\otimes y \in H_1\otimes H_2$$

On using (2) and (3) the above equation becomes

$$A(x,x/\xi)_1(y,y/\eta)_2 \le \sum_i |(x,x_i/\xi)_1|^2 \sum_j |(y,y_j/\eta)_2|^2 \le B(x,x/\xi)_1(y,y/\eta)_2 .$$

This gives $\dfrac{A(y,y/\eta)_2}{\sum_j |(y,y_j/\eta)_2|^2}(x,x/\xi)_1 \le \sum_i |(x,x_i/\xi)_1|^2 \le \dfrac{B(y,y/\eta)_2}{\sum_j |(y,y_j/\eta)_2|^2}(x,x/\xi)_1$.

That is $A_1(x,x/\xi)_1 \le \sum_i |(x,x_i/\xi)|^2 \le B_1(x,x_i/\xi)_1$, for all $x \in H_1$.

Therefore $A_1\|x,\xi\|^2 \le \sum_i |(x,x_i/\xi)_1|^2 \le B_1\|x,\xi\|^2$, for all $x \in H_1$,

where $A_1 = \dfrac{A(y,y/\eta)_2}{\sum_j |(y,y_j/\eta)_2|^2}$ and $B_1 = \dfrac{B(y,y/\eta)_2}{\sum_j |(y,y_j/\eta)_2|^2}$.

Which shows that $\{x_i\}$ is a 2-frame for H_1 associated to ξ. Similarly we can prove that $\{y_j\}$ is a 2-frame for H_2 associated to η.

Conversely, assume that $\{x_i\}$ is a 2-frame for H_1 associated to ξ with frame bounds A_1, B_1 and $\{y_j\}$ is a 2-frame for H_2 associated to η with frame bounds A_2, B_2. Then

$$A_1\|x,\xi\|_1^2 \le \sum_i |(x,x_i/\xi)_1|_1^2 \le B_1\|x,\xi\|_1^2, \text{ for all } x \in H_1 \tag{4}$$

and

$$A_2\|y,\eta\|_2^2 \le \sum_j |(y,y_i/\eta)_2|_2^2 \le B_2\|y,\eta\|_2^2, \text{ for all } y \in H_2 \tag{5}$$

multiplying the Equations (4) and (5) we get

$$A_1A_2\|x\otimes y,\xi\otimes\eta\|^2 \le \sum_{i,j} |(x\otimes y, x_i\otimes y_j/\xi\otimes\eta)|^2 \le B_1B_2\|x\otimes y,\xi\otimes\eta\|^2, \text{ for all } x\otimes y \in H_1\otimes H_2$$

Which shows that $\{x_i\otimes y_j\}$ is a tensor product of frame for $H_1\otimes H_2$. □

Hence we can have the following remark.

Remark 4.3. If the sequences $\{x_i\}$, $\{y_j\}$ and $\{x_i\otimes y_j\}$ are the 2-frames for the Hilbert spaces H_1, H_2 and $H_1\otimes H_2$ respectively and S_ξ, S_η and $S_{\xi\otimes\eta}$ are the frame operators respectively of above frames, then from 3.5, we have the following.

$$S_\xi x = \sum_i (x,x_i/\xi)x_i, \quad S_\eta x = \sum_j (y,y_j/\eta)y_j$$

$$S_{\xi\otimes\eta}(x\otimes y) = \sum_{,i,j} (x\otimes y, x_i\otimes y_j/\xi\otimes\eta)(x_i\otimes y_j), \quad x\in H_1, y\in H_2, x\otimes y\in H_1\otimes H_2$$

Theorem 4.4. If $\{x_i\}$, $\{y_j\}$ and $\{x_i\otimes y_j\}$ are the frames for the Hilbert spaces H_1, H_2 and $H_1\otimes H_2$ with the frame operators S_ξ, S_η and $S_{\xi\otimes\eta}$ respectively, then $S_{\xi\otimes\eta} = S_\xi \otimes S_\eta$.

Proof. For $x\otimes y \in H_1\otimes H_2$, we have

$$\begin{aligned} S_{\xi\otimes\eta}(x\otimes y) &= \sum_{i,j} (x\otimes y, x_i\otimes y_j/\xi\otimes\eta)(x_i\otimes y_j) \\ &= \sum_{i,j} (x,x_i/\xi)_1 (y,y_j/\eta)_2 (x_i\otimes y_j) \\ &= \sum_i (x,x_i/\xi)_1 x_i \otimes \sum_j (y,y_j/\eta)_2 y_j \\ &= S_\xi x\otimes S_\eta y = (S_\xi\otimes S_\eta)(x\otimes y) \end{aligned}$$

Hence $S_{\xi\otimes\eta} = S_\xi\otimes S_\eta$. □

The following two theorems are the extension of 3.6 and 3.7 to the sequence $\{x_i \otimes y_j\}$ so, proofs are left to the reader.

Theorem 4.5. Assume that $\{x_i \otimes y_j\}$ is a sequence in a Hilbert space $H_1 \otimes H_2$. Then $x \otimes y = \sum_{i,j} \left(x \otimes y, x_i \otimes y_j / \xi \otimes \eta\right)\left(x_i \otimes y_j\right)$, $x \otimes y \in H_1 \otimes H_2$ if and only if $\{x_i \otimes y_j\}$ is a 2-normalized tight frame for $H_1 \otimes H_2$.

Theorem 4.6. Suppose that $\{x_i \otimes y_j\}$ is a tensor product of 2-frame for $H_1 \otimes H_2$, and $T_1 \otimes T_2$ is co-isometry. Then $\{(T_1 \otimes T_2)(x_i \otimes y_j)\}$ is a tensor product of 2-frame for $H_1 \otimes H_2$.

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