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Boundedness of Fractional Integral with Variable Kernel and Their Commutators on Variable Exponent Herz Spaces

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Abstract

In this paper, we study the boundedness of the fractional integral operator and their commutator on Herz spaecs with two variable exponents $p(\cdot),q(\cdot)$. By using the properties of the variable exponents Lebesgue spaces, the boundedness of the fractional integral operator and their commutator generated by Lipschitz function is obtained on those Herz spaces.

Keywords

Fractional Integral, Variable Kernel, Commutator, Variable Exponent, Lipschitz Space, Herz Spaces

1. Introduction

Let $0<\mu<n$, $\Omega\in L^{\infty}\left(\mathbb{R}^n\right)\times L^{r}\left(S^{n-1}\right)(r\geq 1)$ is homogenous of degree zero on $\mathbb{R}^n$, S^{n-1} denotes the unit sphere in $\mathbb{R}^n$. If

(i) For any $x,z\in\mathbb{R}^n$, one has $\Omega(x,\lambda z)=\Omega(x,z)$;

(ii) $\|\Omega\|_{L^{\infty}\left(\mathbb{R}^n\right)\times L^{r}\left(S^{n-1}\right)}:=\sup\limits_{x\in\mathbb{R}^n}\left(\int_{S^{n-1}}\left|\Omega(x,z')\right|^{r}\mathrm{d}\sigma(z')\right)^{\frac{1}{r}}<\infty$

The fractional integral operator with variable kernel $T_{\Omega,\mu}$ is defined by

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$$T_{\Omega,\mu} f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} f(y)\,\mathrm{d}y,$$

The commutators of the fractional integral is defined by

$$\left[b^m, T_{\Omega,\mu}\right] f(x) = \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}} \left(b(x)-b(y)\right)^m f(y)\,\mathrm{d}y,$$

When $\mu \equiv 1$, the above integral takes the Cauchy principal value. At this time $\mu \equiv 0$, $T_{\Omega,\mu}$ is much more close related to the elliptic partial equations of the second order with variable coefficients. Now we need the further assumption for $\Omega(x,z)$. It satisfies

$$\int_{s^{n-1}} \Omega(x, z')\,\mathrm{d}\sigma(z') = 0, \forall x \in \mathbb{R}^n$$

For $r \geq 1$, we say Kernel function $\Omega(x,z)$ satisfies the L^r-Dini condition, if Ω meets the conditions (i), (ii) and

$$\int_0^1 \frac{\omega_r(\delta)}{\delta}\,\mathrm{d}\delta < \infty$$

where $\omega_r(\delta)$ denotes the integral modulus of continuity of order *r* of Ω defined by

$$\omega_r(\delta) = \sup_{x\in\mathbb{R}^n, |\rho|<\delta} \left(\int_{s^{n-1}} \left|\Omega(x, \rho z') - \Omega(x, z')\right|^r \mathrm{d}\sigma(z')\right)^{\frac{1}{r}}$$

where ρ is the a rotation in $\mathbb{R}^n$

$$|\rho| = \sup_{z' \in S^{n-1}} |\rho z' - z'|$$

when $\Omega \equiv 1$, $T_{\Omega,\mu}$ is the fraction integral operator

$$T_{\Omega,\mu} f(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\mu}}\,\mathrm{d}y.$$

The corresponding fractional maximal operator with variable kernel is defined by

$$M_{\Omega,\mu} f(x) = \sup_{r>0} \frac{1}{r^{n-\mu}} \int_{|x-y|<r} \left|\Omega(x, x-y)\right|\mathrm{d}y.$$

We can easily find that when $\Omega \equiv 1$ M_{Ω}, μ is just the fractional maximal operator

$$M_{\Omega,\mu} f(x) = \sup_{r>0} \frac{1}{r^{n-\mu}} \int_{|x-y|<r} |f(y)|\,\mathrm{d}y$$

Especially, in the case $\mu \equiv 0$, the fractional maximal operator reduces the Hardy-Littelewood maximal operator.

Many classical results about the fractional integral operator with variable kernel have been achieved [1]-[5]. In 1971, Muckenhoupt and Wheeden [6] had proved the operator $T_{\Omega,\mu}$ was bounded from L^p to L^q. In 1991, Kováčik and Rákosník [7] introduced variable exponents Lebesgue and Sobolev spaces as a new method for dealing with nonlinear Dirichet boundary value problem. In the last 20 years, more and more researchers have been interested in the theory of the variable exponent function space and its applications [8]-[14]. In 2012, Wu Huiling and Lan Jiacheng [15] proved the bonudedness property of $T_{\Omega,\mu}$ with a rough kernel on variable exponents Lebesgue spaces.

Recently, Wang and Tao [16] introduced the class of Herz spaces with two variable exponents, and also studied the Parameterized Littlewood-Paley operators and their commutators on Herz spaces with variable exponents.

The main purpose of this paper is to discuss the boundedness of the fractional integral with variable kernel $T_{\Omega,\mu}$ and their commutators $\left[b^m, T_{\Omega,\mu}\right]$ are bonuded on Herz spaces with two variable exponents or not.

Throughout this paper $|E|$ denotes the Lebesgue measure, χ_E means he characteristic function of a measurable set $S \subset \mathbb{R}^n$. C always means a positive constant independent of the main parameters and may change from one occurrence to another.

2. Definition of Function Spaces with Variable Exponent

In this section we define the Lebesgue spaces with variable exponent and Herz spaces with two variable exponent, and also define the mixed Lebesgue sequence spaces.

Let E be a measurable set in $\mathbb{R}^n$ with $|E| > 0$. We first define the Lebesgue spaces with variable exponent.

Definition 2.1. see [1] Let $p(\cdot): E \to [1, \infty)$ be a measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}(E)$ is defined by

$$L^{p(\cdot)}(E) = \left\{ f \text{ is measurable} : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x < \infty \text{ for some constant } \eta > 0 \right\}$$

The space $L^{p(\cdot)}_{loc}(E)$ is defined by

$$L^{p(\cdot)}_{loc}(E) = \left\{ f \text{ is measurable} : f \in L^{p(\cdot)}(K) \text{ for all compact } K \subset E \right\}$$

The Lebesgue spaces $L^{p(\cdot)}(E)$ is a Banach spaces with the norm defined by

$$\|f\|_{L^{p(\cdot)}(E)} = \inf \left\{ \eta > 0 : \int_E \left(\frac{|f(x)|}{\eta} \right)^{p(x)} \mathrm{d}x \le 1 \right\}$$

We denote

$$p_- = \operatorname{essinf} \{ p(x) : x \in E \}, \quad p_+ = \operatorname{esssup} \{ p(x) : x \in E \}.$$

Then $\mathcal{P}(E)$ consists of all $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let M be the Hardy-Littlewood maximal operator. We denote $\mathfrak{B}(E)$ to be the set of all function $p(\cdot) \in \mathcal{P}(E)$ satisfying the M is bounded on $L^{p(\cdot)}(E)$.

Definition 2.2. see [17] Let $p(\cdot), q(\cdot) \in \mathcal{P}(E)$. The mixed Lebesgue sequence space with variable exponent $l^{q(\cdot)}\left(L^{p(\cdot)}\right)$ is the collection of all sequences $\{f_j\}_{j=0}^{\infty}$ of the measurable functions on $\mathbb{R}^n$ such that

$$\left\| \{f_j\}_{j=0}^{\infty} \right\|_{l^{q(\cdot)}\left(L^{p(\cdot)}\right)} = \inf \left\{ \eta > 0 : Q_{l^{q(\cdot)}\left(L^{p(\cdot)}\right)} \left(\left\{ \frac{f_j}{\mu} \right\}_{j=0}^{\infty} \right) \le 1 \right\} < \infty$$

$$Q_{l^{q(\cdot)}\left(L^{p(\cdot)}\right)} \left(\{f_j\}_{j=0}^{\infty} \right) = \sum_{j=0}^{\infty} \inf \left\{ \mu_j ; \int_{R^n} \left(\frac{|f_j(x)|}{\mu_j^{\frac{1}{q(x)}}} \right)^{p(x)} \mathrm{d}x \le 1 \right\}$$

Noticing $q_+ < \infty$, we see that

$$Q_{l^{q(\cdot)}\left(L^{p(\cdot)}\right)} \left(\{f_j\}_{j=0}^{\infty} \right) = \sum_{j=0}^{\infty} \left\| |f_j|^{q(\cdot)} \right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}$$

Let $B_k = \left\{ x \in \mathbb{R}^n : |x| \le 2^k \right\}, C_k = B_k \setminus B_{k-1}, \chi_k = \chi_{C_k}, k \in \mathbb{Z}$

Definition 2.3. see [16] Let $\alpha \in \mathbb{R}, p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. The homogeneous Herz space with variable exponent $\dot{K}^{\alpha, q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$ is defined by

$$\dot{K}^{\alpha, q(\cdot)}_{p(\cdot)}(\mathbb{R}^n) = \left\{ f \in L^{p(\cdot)}_{Loc}\left(\mathbb{R}^n \setminus \{0\}\right) : \|f\|_{\dot{K}^{\alpha, q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} < \infty \right\}$$

where

$$\|f\|_{\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)} = \left\|\left\{2^{k\alpha}|f\chi_k|\right\}_{k=0}^{\infty}\right\|_{l^{q(\cdot)}(L^{p(\cdot)})} = \inf\left\{\eta>0:\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q(\cdot)}}}\le 1\right\}$$

Remark 2.1. see [16] (1) If $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfying $(q_1)_+\le(q_2)_+$, then

$$\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)\subset\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$$

(2) If $q_1(\cdot),q_2(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ and $(q_1)_+\le(q_2)_+$, then $\frac{q_2(\cdot)}{q_1(\cdot)}\in\mathcal{P}(\mathbb{R}^n)$ and $\frac{q_2(\cdot)}{q_1(\cdot)}\ge 1$. Thus, by Lemma 3.7 and Remark 2.2, for any $f\in\dot{K}^{\alpha,q(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$, we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p(\cdot)}{q_2(\cdot)}}}\le\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|^{p_h}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\le\left\{\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|f\chi_k|}{\eta}\right)^{q_1(\cdot)}\right\|^{p_h}_{L^{\frac{p(\cdot)}{q_1(\cdot)}}}\right\}^{p_*}\le 1$$

where

$$p_h=\begin{cases}\left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_-, & \frac{2^{k\alpha}|f\chi_k|}{\eta}\le 1\\ \left(\frac{q_2(\cdot)}{q_1(\cdot)}\right)_+, & \frac{2^{k\alpha}|f\chi_k|}{\eta}>1\end{cases}$$

$$p_*=\begin{cases}\min_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty}a_h\le 1\\ \max_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty}a_h>1\end{cases}$$

This implies that $\dot{K}^{\alpha,q_1(\cdot)}_{p(\cdot)}(\mathbb{R}^n)\subset\dot{K}^{\alpha,q_2(\cdot)}_{p(\cdot)}(\mathbb{R}^n)$.

Remark 2.2. Let $h\in\mathbb{N}, a_h\ge 0, 1\le p_h<\infty$. then

$$\sum_{h=0}^{\infty}a_h\le\left(\sum_{h=0}^{\infty}a_h\right)^{p_*}$$

where

$$p_*=\begin{cases}\min_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty}a_h\le 1\\ \max_{h\in\mathbb{N}} p_h, & \sum_{h=0}^{\infty}a_h>1\end{cases}$$

Definition 2.4. see [18] For $0<\beta\le 1$, the Lipschitz space $Lip_\beta(\mathbb{R}^n)$ is defined by

$$Lip_\beta(\mathbb{R}^n)=\left\{f:\|f\|_{Lip_\beta}=\sup_{x,y\in\mathbb{R}^n,x\ne y}\frac{|f(x)-f(y)|}{|x-y|^\beta}<\infty\right\} \tag{1.1}$$

3. Properties of Variable Exponent

In this section we state some properties of variable exponent belonging to the class $\mathfrak{B}(\mathbb{R}^n)$ and $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$.

Proposition 3.1. see [1] If $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$ satisfies

$$|p(x)-p(y)|\le\frac{-C}{\log(|x-y|)}, |x-y|\le 1/2$$

$$\left|p(x)-p(y)\right| \le \frac{C}{\log\left(e+|x|\right)}, |y| \ge |x|$$

then, we have $p(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$.

Proposition 3.2. see [15] Suppose that $p_1(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$, $\Omega \in L^\infty\left(\mathbb{R}^n\right) \times L^r\left(S^{n-1}\right)$. Let $0<\mu \le \frac{n}{\left(p_1\right)_+}$, and define the variable exponent $p_2(\cdot)$ by: $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\mu}{n}$. Then we have that for all $f \in L^{p_1(\cdot)}\left(\mathbb{R}^n\right)$,

$$\left\|T_{\Omega,\mu} f\right\|_{L^{p_2(\cdot)}\left(\mathbb{R}^n\right)} \le c\|f\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}$$

Proposition 3.3. Suppose that $p_1(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$, $b \in Lip_\beta\left(\mathbb{R}^n\right)$, $0<\beta \le 1$, $\Omega \in L^\infty\left(\mathbb{R}^n\right) \times L^r\left(S^{n-1}\right)$. Let $0<\mu+m\beta \le \frac{n}{\left(p_1\right)_+}$, and define the variable exponent $p_2(\cdot)$ by: $\frac{1}{p_1(x)}-\frac{1}{p_2(x)}=\frac{\mu+m\beta}{n}$. Then

$$\left\|\left[b^m, T_{\Omega,\mu}\right] f\right\|_{L^{p_2(\cdot)}\left(\mathbb{R}^n\right)} \le C\|b\|_{Lip_\beta}^m\|f\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}$$

Proof

$$\begin{aligned}
\left[b^m, T_{\Omega,\mu}\right] f(x) &= \int_{\mathbb{R}^n} \frac{\Omega(x, x-y)}{|x-y|^{n-\mu}}(b(x)-b(y))^m f(y) \mathrm{d}y \\
&\le \int_{\mathbb{R}^n} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\mu}}\left|(b(x)-b(y))\right|^m |f(y)| \mathrm{d}y \\
&\le C\|b\|_{Lip_\beta} \int_{\mathbb{R}^n} \frac{|\Omega(x, x-y)|}{|x-y|^{n-\mu-m\beta}}|f(y)| \mathrm{d}y \\
&\le C\|b\|_{Lip_\beta} T_{|\Omega|,\mu+m\beta}(|f|)
\end{aligned}$$

By Proposition 3.2, we get

$$\left\|\left[b^m, T_{\Omega,\mu}\right] f\right\|_{L^{p_2(\cdot)}\left(\mathbb{R}^n\right)} \le C\|b\|_{Lip_\beta}\left\|T_{|\Omega|,\mu+m\beta}(|f|)\right\|_{L^{p_2(\cdot)}} \le C\|b\|_{Lip_\beta}^m\|f\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}.$$

Now, we need recall some lemmas

Lemma 3.1. see [13] Given $p(\cdot): \mathbb{R}^n \to [1, \infty)$ have that for all function *f* and *g*,

$$\int_{\mathbb{R}^n}|f(x) g(x)| \mathrm{d}x \le C\|f\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)}\|g\|_{L^{p'(\cdot)}\left(\mathbb{R}^n\right)}$$

Lemma 3.2. see [19] Suppose that $0<\mu<n$, $r>1$, $\Omega \in L^\infty\left(\mathbb{R}^n\right) \times L^r\left(S^{n-1}\right)$ satisfies the L^r-Dini condition. If there exists an $0<\alpha_0<1/2$ such that $|y|<\alpha_0 R$ then

$$\left(\int_{R<|x|<2R}\left|\frac{\Omega(x, x-y)}{|x-y|^{n-\mu}}-\frac{\Omega(x, x)}{|x|^{n-\mu}}\right|^r \mathrm{d}x\right)^{\frac{1}{r}} \le C R^{\left(\frac{n}{r}-n+\mu\right)}\left(\frac{|y|}{R}+\int_{|y|/2R}^{|y|/R} \frac{\omega_r(\delta)}{\delta} \mathrm{d}\delta\right)$$

Lemma 3.3. see [20] Suppose that $x \in \mathbb{R}^n$, the variable function $\tilde{q}(x)$ is defined by $\frac{1}{p(x)}=\frac{1}{q}+\frac{1}{\tilde{q}(x)}$, then for all measurable function *f* and *g*, we have

$$\|f(x) g(x)\|_{L^{p(\cdot)}\left(\mathbb{R}^n\right)} \le C\|g(x)\|_{L^q\left(\mathbb{R}^n\right)}\|f(x)\|_{L^{\tilde{q}(\cdot)}\left(\mathbb{R}^n\right)}$$

Lemma 3.4. see [21] Suppose that $p(\cdot) \in \mathfrak{B}\left(\mathbb{R}^n\right)$ and $0<p^- \le p^+<\infty$.

1) For any cube and $|Q| \le 2^n$, all the $\chi \in Q$, then: $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p(x)}$

2) For any cube and $|Q| \ge 1$, then $\|\chi_Q\|_{L^{p(\cdot)}} \approx |Q|^{1/p_\infty}$ where

$$p_\infty = \lim_{x\to\infty} p(x)$$

Lemma 3.5. see [22] If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, then there exist constants $\delta_1, \delta_2, C > 0$ such that for all balls B in $\mathbb{R}^n$ and all measurable subset $S \subset R$

$$\frac{\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}}{\|\chi_S\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \le C\frac{|B|}{|S|}, \quad \frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_1},$$

$$\frac{\|\chi_S\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}}{\|\chi_B\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}} \le C\left(\frac{|S|}{|B|}\right)^{\delta_2}$$

Lemma 3.6. see [13] If $p(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$, there exist a constant $C > 0$ such that for any balls B in $\mathbb{R}^n$. we have

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}\|\chi_B\|_{L^{p'(\cdot)}(\mathbb{R}^n)} \le C$$

Lemma 3.7. see [16] Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$. If $f \in L^{p(\cdot)q(\cdot)}$, then

$$\min\left(\|f\|_{L^{p(\cdot)q(\cdot)}}^{q_+}, \|f\|_{L^{p(\cdot)q(\cdot)}}^{q_-}\right) \le \left\||f|^{q(\cdot)}\right\|_{L^{p(\cdot)}} \le \max\left(\|f\|_{L^{p(\cdot)q(\cdot)}}^{q_+}, \|f\|_{L^{p(\cdot)q(\cdot)}}^{q_-}\right)$$

4. Main Theorems and Their Proof

Theorem 1. Suppose that $0 < \mu < n$, $\mu - n\delta_2 < \alpha < n\delta_1$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r > p_2^+)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. And let $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfy $0 < \mu \le \frac{n}{(p_1)_+}$ and define the variable exponent $p_2(x)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu}{n}$. Then the operators $T_{\Omega,\mu}$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

Theorem 2. Let $b \in Lip_\beta(\mathbb{R}^n), m \in \mathbb{N}$. Suppose that $0 < \mu < n, (\mu + m\beta) - n\delta_2 < \alpha < n\delta_1$, $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})(r > p_2^+)$, $q_1(\cdot), q_2(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $(q_2)_- \ge (q_1)_+$. If $p_1(\cdot) \in \mathfrak{B}(\mathbb{R}^n)$ satisfy $0 < \mu + m\beta \le \frac{n}{(p_1)_+}$ and define the variable exponent $p_2(x)$ by $\frac{1}{p_1(x)} - \frac{1}{p_2(x)} = \frac{\mu + m\beta}{n}$. Then the commutators $[b^m, T_{\Omega,\mu}]$ is bounded from $\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$ to $\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)$.

Proof of Theorem1:

Let $f \in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_j = \sum_{j=-\infty}^{\infty} f_j(x)$$

From definition of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$

$$\|T_{\Omega,\mu}, f\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}|T_{\Omega,\mu}(f)\chi_k|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le 1\right\}$$

Since

$$\left\|\left(\frac{2^{k\alpha}\left|T_{\Omega,\mu}(f)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{\infty}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_{11}+\eta_{12}+\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_{11}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_{12}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_{13}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

where

$$\eta_{11}=\left\|\left\{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}(f_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}\left(L^{p_2(\cdot)}\right)}$$

$$\eta_{12}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}(f_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}\left(L^{p_2(\cdot)}\right)}$$

$$\eta_{13}=\left\|\left\{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}(f_j)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}\left(L^{p_2(\cdot)}\right)}$$

And $\eta=\eta_{11}+\eta_{12}+\eta_{13}$, thus

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq C$$

That is

$$\left\|T_{\Omega,\mu}(f)\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)}\leq C\eta=C\left[\eta_{11}+\eta_{12}+\eta_{13}\right].$$

This implies only to prove $\eta_{11},\eta_{12},\eta_{13}\leq C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$. Denote $\eta_1=\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$

Now we consider η_{12}. Applying Lemma 3.7

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_1}\right\|_{L^{p_2(\cdot)}}^{\left(q_2^1\right)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{k+1}\left\|\frac{2^{k\alpha}\left|T_{\Omega,\mu}(f_j)\chi_k\right|}{\eta_1}\right\|_{L^{p_2(\cdot)}}\right)^{\left(q_2^1\right)k}$$

where

$$\left(q_2^1\right)k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le 1\\ \left(q_2\right)_+, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

By the Proposition 3.2, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{k+1}\left\|\frac{2^{k\alpha}\left|f_j\right|}{\eta_1}\right\|_{L^{p_1(\cdot)}}^{\left(q_2^1\right)k}$$

Since $f\in\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)$, then we have $\left\|\dfrac{2^{j\alpha}\left|f\chi_j\right|}{\eta_1}\right\|_{L^{p(\cdot)}}\le 1$, and

$$\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}\left|f\chi_j\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\le 1$$

By Lemma 3.7 and Remark 2.2, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|f\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}^{\frac{\left(q_2^1\right)k}{\left(q_1\right)_+}}$$

$$\le C\left[\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|f\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\right]^{q_*}\le C$$

Hence $\left(p_1\right)_+,\left(p_2\right)_-\le\left(q_2^1\right)k$, and $q_*=\min_{k\in\mathbb{N}}\dfrac{\left(q_2^1\right)k}{\left(q_1\right)_+}$, this implies that

$$\eta_{12}\le C\eta_1\le C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)}$$

Now, we estimate of η_{11} using size condition of f_j and Minkowski inequality, when $j\le k-1$ we get,

$$\left\|T_{\Omega,\mu}\left(f_j\right)\chi_k\right\|_{L^{p_2(\cdot)}\left(\mathbb{R}^n\right)}\le\int_{B_j}f_j\left(y\right)\left\|\left|\frac{\Omega\left(x,x-y\right)}{|x-y|^{n-\mu}}-\frac{\Omega\left(x,x\right)}{|x|^{n-\mu}}\right|\chi_k\right\|_{L^{p_2(\cdot)}\left(\mathbb{R}^n\right)}\mathrm{d}y$$

Since $r > p_2^+$ we define the variable exponent $\frac{1}{p_2(x)} = \frac{1}{r} + \frac{1}{\tilde{p}_2(x)}$, by Lemma 3.3 we get

$$\left\| \left| \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x,x)}{|x|^{n-\mu}} \right| \chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le \left\| \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x,x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \|\chi_k\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)}$$
$$\le \left\| \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x,x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \|\chi_{B_k}\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)}$$

According Lemma 3.4 and the formula $\frac{1}{\tilde{p}_2(x)} = \frac{1}{p_2(x)} - \frac{1}{r}$, then we have

$$\|\chi_{B_k}\|_{L^{\tilde{p}_2(x)}(\mathbb{R}^n)} \approx \|\chi_{B_k}\|_{L^{p_2(x)}(\mathbb{R}^n)} |B_k|^{\frac{-1}{r}} \approx \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} |B_k|^{\frac{-1}{r}-\frac{\mu}{n}} \tag{1.2}$$

By Lemma 3.2, we get

$$\left\| \frac{\Omega(x,x-y)}{|x-y|^{n-\mu}} - \frac{\Omega(x,x)}{|x|^{n-\mu}} \right\|_{L^r(\mathbb{R}^n)} \le CR^{\left(\frac{n}{r}-n+\mu\right)} \left(\frac{|y|}{2^{k-1}} + \int_{|y|/2^k}^{|y|/2^{k-1}} \frac{\omega_r(\delta)}{\delta} \mathrm{d}\delta \right)$$
$$\le CR^{\left(\frac{n}{r}-n+\mu\right)} 2^{(j-k)\beta} \left(1 + \int_0^1 \frac{\omega_r(\delta)}{\delta} \mathrm{d}\delta \right)$$
$$\le C\, 2^{(k-1)\left(\frac{n}{r}-n+\mu\right)}$$

It follows that

$$\left\| T_{\Omega,\mu}(f_j)\chi_k \right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)} \le C 2^{-kn} \int_{B_j} f_j(y)\,\mathrm{d}y \|\chi_{B_k}\|_{L^{p_1(x)}(\mathbb{R}^n)} \tag{1.3}$$

By the Equation (1.3) and using Lemmas 3.1, 3.5, 3.6, 3.7, we can obtain

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} T_{\Omega,\mu}(f_j)\chi_k \right|}{\eta_1} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le C \sum_{k=-\infty}^{\infty} \left\| \frac{2^{k\alpha} \left| \sum_{j=-\infty}^{k-2} T_{\Omega,\mu}(f_j)\chi_k \right|}{\eta_1} \right\|_{L^{p_2(\cdot)}}^{\left(q_2^2\right)k}$$
$$\le C \sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{-kn} \left\| \frac{f_j}{\eta_1} \right\|_{L^1(\mathbb{R}^n)} \|\chi_k\|_{L^{p_1(\cdot)}} \right]^{\left(q_2^2\right)k} \le C \sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \sum_{j=-\infty}^{k-2} \left\| \frac{f_j}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} 2^{-kn} \|\chi_{B_k}\|_{L^{p_1(\cdot)}} \right]^{\left(q_2^2\right)k}$$
$$\le C \sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \sum_{j=-\infty}^{k-2} \left\| \frac{f_j}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \frac{\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}}{\|\chi_{B_k}\|_{L^{p_1'(\cdot)}}} \right]^{\left(q_2^2\right)k} \le C \sum_{k=-\infty}^{\infty} \left[2^{k\alpha} \sum_{j=-\infty}^{k-2} 2^{(j-k)n\delta_1} \left\| \frac{f_j}{\eta_1} \right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)} \right]^{\left(q_2^2\right)k}$$
$$\le C \sum_{k=-\infty}^{\infty} \left[\sum_{j=-\infty}^{k-2} 2^{(j-k)(n\delta_1-\alpha)} \left\| \left(\frac{\left|2^{\alpha j} f_j \chi_k\right|}{\eta_1} \right)^{q_1(\cdot)} \right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}} \right]^{\left(q_2^2\right)k}$$

where

$$\left(q_2^2\right)k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le 1\\ \left(q_2\right)_+, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

Since $f\in\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)$, then we have $\left\|\dfrac{2^{j\alpha}\left|f\chi_j\right|}{\eta_1}\right\|_{L^{p(\cdot)}}\le 1$, and

$$\sum_{j=-\infty}^{\infty}\left\|\left(\frac{2^{j\alpha}\left|f\chi_j\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}}\le 1$$

Now if $\left(q_1\right)_+<1$, then we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} &\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1-\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}\right]^{\frac{\left(q_2^2\right)k}{\left(q_1\right)_+}}\\ &\le C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_1-\alpha)}\right]^{q_*}\le C\end{aligned}$$

where $q_*=\min_{k\in\mathbb{N}}\dfrac{\left(q_2^2\right)k}{\left(q_1\right)_+}$

If $\left(q_1\right)_+\ge 1$, then we have

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\\ &\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1-\alpha)\frac{\left(q_2\right)_+}{2}}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}\right]^{\frac{\left(q_2^2\right)k}{\left(q_1\right)_+}}\times\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1-\alpha)\frac{\left(\left(q_1\right)_+\right)'}{2}}\right]^{\frac{\left(q_2^2\right)_+}{\left(\left(q_1\right)_+\right)'}}\\ &\le C\left[\sum_{j=-\infty}^{\infty}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}\sum_{k=j+2}^{\infty}2^{(j-k)(n\delta_1-\alpha)\frac{\left(q_1\right)_+}{2}}\right]^{q_*}\le C\end{aligned}$$

where $q_* = \min_{k\in\mathbb{N}} \frac{\left(q_2^2\right)k}{\left(q_1\right)_+}$, this implies that

$$\eta_{11} \le C\eta_1 \le C\|f\|_{\dot{K}^{\alpha,q_1(\cdot)}_{p_1(\cdot)}\left(\mathbb{R}^n\right)}$$

Finally, we estimate η_{13} by Lemma 3.7, we get

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le C\sum_{k=-\infty}^{\infty}\left\|\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right\|_{L^{p_2(\cdot)}}^{\left(q_2^3\right)k}\\ &\le \sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}\left\|\frac{1}{\eta_1}\left|\int_{C_j}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f_j(y)\mathrm{d}y\,\chi_k\right|\right\|_{L_{p_2(\cdot)}}\right]^{\left(q_2^3\right)k}\end{aligned} \tag{1.4}$$

Note that, when $x\in C_k$, $j\ge k+2$, then $|x-y|\sim|x|$. Therefore, applying the generalized Hölder's Inequality, we have

$$\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f_j(y)\right|\mathrm{d}y \le \left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}\chi_j\right\|_{L^{p_1'(\cdot)}}$$

Define the variable exponent $\frac{1}{p_1'(\cdot)} = \frac{1}{r} + \frac{1}{\widetilde{p_1'}(\cdot)}$ by Lemma 3.3, then we have

$$\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f_j(y)\right|\mathrm{d}y \le \left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\Omega(x,x-y)\right\|_{L^r}\left\|\frac{\chi_j}{|x-y|^{n-\mu}}\right\|_{L^{\widetilde{p_1'}(\cdot)}}$$

$$\begin{aligned}&\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f_j(y)\right|\mathrm{d}y\\ &\le C2^{-j(n-\mu)}\left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\chi_j\right\|_{L^{\widetilde{p_1'}(\cdot)}}\left[\int_{2^{j-2}}^{2^j}r^{n-1}\mathrm{d}r\left(\int_{S^{n-1}}\left|\Omega(x,y')\right|^r\mathrm{d}\sigma(y')\right)\right]^{\frac{1}{r}}\\ &\le C2^{-j(n-\mu)}\left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\chi_j\right\|_{L^{\widetilde{p_1'}(\cdot)}}2^{\frac{jn}{r}}\left\|\Omega\right\|_{L^\infty\left(\mathbb{R}^n\right)\times L^r\left(S^{n-1}\right)}\\ &\le C2^{-j(n-\mu)}\left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\chi_j\right\|_{L^{\widetilde{p_1'}(\cdot)}}2^{\frac{jn}{r}}\end{aligned}$$

According Lemma 3.4 and the formula $\frac{1}{\widetilde{p_1'}(\cdot)} = \frac{1}{p_1'(\cdot)} - \frac{1}{r}$, we have $\left\|\chi_{B_j}\right\|_{L^{\widetilde{p_1'}(\cdot)}} \approx \left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}\left|B_j\right|^{\frac{-1}{r}}$ Then we get

$$\int_{C_j}\left|\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}f_j(y)\right|\mathrm{d}y \le 2^{-j(n-\mu)}\left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}} \tag{1.5}$$

From Equations (1.4), (1.5) and using Lemma 3.7, and $\left\|\left(\frac{2^{j\alpha}\left|f\chi_j\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{\frac{p_1(\cdot)}{q_1(\cdot)}}} \le 1$ we can obtain

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j(n-\mu)}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}\left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}}\right]^{\left(q_2^3\right)k}$$

Note that

$\left\|\chi_{B_k}\right\|_{L^{p_2(\cdot)}}\leq C2^{-k\mu}\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}}$ see [9].

Then we have

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{j\mu}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}2^{-jn}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}2^{-k\mu}\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}}\right]^{\left(q_2^3\right)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(j-k)\mu}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}\frac{\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}}}{\left\|\chi_{B_j}\right\|_{L^{p_1(\cdot)}}}\right]^{\left(q_2^3\right)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-\mu)}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}\left(\mathbb{R}^n\right)}\right]^{\left(q_2^3\right)k}$$

$$\leq C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-\mu+\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}^{\frac{1}{(q_1)_+}}\right]^{\left(q_2^3\right)k}$$

where

$$\left(q_2^3\right)k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\leq 1\\ \left(q_2\right)_+, & \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}T_{\Omega,\mu}\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

Since $\left(q_2\right)_-\geq\left(q_1\right)_+$ and $\alpha>\mu-n\delta_2$, as the same η_{11} we have

$$\eta_{13} \le C\eta_1 \le C\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$$

This completes the proof Theorem 1.

Proof of Theorem 2

Let $b \in Lip_\beta(\mathbb{R}^n)$, $f \in \dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)$. We write

$$f(x) = \sum_{j=-\infty}^{\infty} f(x)\chi_k = \sum_{j=-\infty}^{\infty} f_j(x)$$

From definition of $\dot{K}_{p(\cdot)}^{\alpha,q(\cdot)}(\mathbb{R}^n)$

$$\left\|\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} = \inf\left\{\eta > 0 : \sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le 1\right\}$$

Since

$$\begin{aligned}
&\left\|\left(\frac{2^{k\alpha}\left|\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&\le \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{\infty}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta_{21}+\eta_{22}+\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta_{21}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \\
&+\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta_{22}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} + \left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|}{\eta_{23}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}
\end{aligned}$$

where

$$\eta_{21} = \left\|\left\{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

$$\eta_{22} = \left\|\left\{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

$$\eta_{23} = \left\|\left\{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right|\right\}_{k=-\infty}^{\infty}\right\|_{l^{q_2(\cdot)}(L^{p_2(\cdot)})}$$

And $\eta = \eta_{21}+\eta_{22}+\eta_{23}$. The similar to prove of Theorem 1

$$\left\|\left[b^m, T_{\Omega,\mu}\right](f)\chi_k\right\|_{\dot{K}_{p_2(\cdot)}^{\alpha,q_2(\cdot)}(\mathbb{R}^n)} \le C\eta \le C\left[\eta_{21}+\eta_{22}+\eta_{23}\right]$$

Hence $\eta_{21}, \eta_{22}, \eta_{23} \le C\|b\|_{Lip_\beta(\mathbb{R}^n)}^m \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$. Denote $\eta_1 = \|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$

First we estimate η_{22}. Note that $\left[b^m, T_{\Omega,\mu}\right]$ is bonuded on $L^{p(\cdot)}(\mathbb{R}^n)$ (Proposition 3.3), similarly to estimate for η_{12} in the proof of the Theorem 1, we get that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k-1}^{k+1}\left[b^m,T_{\Omega,\mu}\right](f_j)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C$$

That is

$$\eta_{22}\le C\eta_1\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\le C\|b\|^m_{Lip_\beta(\mathbb{R}^n)}\|f\|_{\dot K^{\alpha,q_1(\cdot)}_{p_1(\cdot)}(\mathbb{R}^n)}$$

Now, we estimate of η_{21}. Using size condition of f_j and Minkowski inequality, when $j\le k-1$ we get,

$$\begin{aligned}\left|\left[b^m,T_{\Omega,\mu}\right]\right|&=\left|\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}(b(x)-b(y))^m f(y)\mathrm{d}y\right|\\&\le\left|\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}|x-y|^{m\beta}\|b\|^m_{Lip_\beta}f(y)\mathrm{d}y\right|\\&\le\|b\|^m_{Lip_\beta}\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\mu n-\mu-m\beta}}|f(y)|\mathrm{d}y\\&\le\|b\|^m_{Lip_\beta}\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-\mu n-(\mu+m\beta)}}f(y)\mathrm{d}y\end{aligned}$$

We have that

$$\left\|\left[b^m,T_{\Omega,\mu}\right]\chi_k\right\|_{L^{p_2(\cdot)}}\le C\|b\|^m_{Lip_\beta}\int_{B_j}f_j(y)\left\|\left|\frac{\Omega(x,x-y)}{|x-y|^{n-(\mu+m\beta)}}-\frac{\Omega(x,x)}{|x|^{n-(\mu+m\beta)}}\right|\chi_k\right\|_{L^{p_2(\cdot)}(\mathbb{R}^n)}\mathrm{d}y \tag{1.6}$$

The similar way to estimate of $T_{\Omega,\mu}$ in the proof of Theorem 1, we get that

$$\left\|\left[b^m,T_{\Omega,\mu}\right]\chi_k\right\|_{L^{p_2(\cdot)}}\le C\|b\|^m_{Lip_\beta}2^{-kn}\int_{B_j}f_j(y)\mathrm{d}y\left\|\chi_{B_k}\right\|_{L^{p_1(x)}(\mathbb{R}^n)} \tag{1.7}$$

By (1.7) and lemma 3.7, we obtain that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}\left[b^m,T_{\Omega,\mu}\right](f_j)\chi_k\right|}{\eta_1\|b\|^m_{Lip_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le C\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}\left[b^m,T_{\Omega,\mu}\right](f_j)\chi_k\right|}{\eta_1\|b\|^m_{Lip_\beta}}\right)^{q_2(\cdot)}\right\|_{L^{p_2(\cdot)}}\\&\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{-kn}\left\|\frac{f_j}{\eta_1}\right\|_{L^1(\mathbb{R}^n)}\left\|\chi_k\right\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)k}\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}2^{-kn}\left\|\chi_{B_k}\right\|_{L^{p_1(\cdot)}}\right]^{(q_2^2)k}\\&\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}\left\|\chi_{B_k}\right\|^{-1}_{L^{p_1'(\cdot)}}\right]^{(q_2^2)k}\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\left\|\chi_{B_j}\right\|_{L^{p_1'(\cdot)}}}{\left\|\chi_{B_k}\right\|_{L^{p_1'(\cdot)}}}\right]^{(q_2^2)k}\\&\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=-\infty}^{k-2}2^{(j-k)n\delta_1}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)k}\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1-\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|^{\frac{1}{(q_1)_+}}_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^2)k}\end{aligned}$$

where

$$\left(q_2^2\right)k=\begin{cases}\left(q_2\right)_-, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=\infty}^{k-2}\left[b^m,T_{\Omega,\mu}\right]\left(f_j\right)\chi_k\right|}{\eta_1\|b\|_{Lip_\beta}^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}\le 1\\ \left(q_2\right)_+, & \left\|\left(\dfrac{2^{k\alpha}\left|\sum_{j=k-1}^{\infty}\left[b^m,T_{\Omega,\mu}\right]\left(f_j\right)\chi_k\right|}{\eta_1\|b\|_{Lip_\beta}^m}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}>1\end{cases}$$

Since $\left(q_2\right)_-\ge\left(q_1\right)_+$ and $\alpha>-n\delta_1$, the similar way to estimate η_{11} in the proof of Theorem1, we can obtain that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=-\infty}^{k-2}\left[b^m,T_{\Omega,\mu}\right]\left(f_j\right)\chi_k\right|}{\eta_1}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=-\infty}^{k-2}2^{(j-k)(n\delta_1-\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}\left(\mathbb{R}^n\right)}\right]^{q_*}\le C$$

where $q_*=\min_{k\in\mathbb{N}}\dfrac{\left(q_2^2\right)k}{\left(q_1\right)_+}$, this implies that

$$\eta_{21}\le C\eta_1\|b\|_{Lip_\beta\left(\mathbb{R}^n\right)}^m\le C\|b\|_{Lip_\beta\left(\mathbb{R}^n\right)}^m\|f\|_{\dot K_{p_1(\cdot)}^{\alpha,q_1(\cdot)}\left(\mathbb{R}^n\right)}$$

Finally, we estimate η_{23}. Note that, when $x\in C_k$, $j\ge k+2$, then $|x-y|\sim|y|$, we can obtain that

$$\left|\left[b^m,T_{\Omega,\mu}\right]\right|=\left|\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}\left(b(x)-b(y)\right)^m f(y)\mathrm{d}y\right|$$

$$\le\left|\int_{\mathbb{R}^n}\frac{\Omega(x,x-y)}{|x-y|^{n-\mu}}|x-y|^{m\beta}\|b\|_{Lip_\beta}^m f(y)\mathrm{d}y\right|$$

$$\le\|b\|_{Lip_\beta}^m\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-(\mu+m\beta)}}f(y)\mathrm{d}y$$

Then we have

$$\left|\left[b^m,T_{\Omega,\mu}\right]\right|\le\|b\|_{Lip_\beta}^m\int_{\mathbb{R}^n}\frac{|\Omega(x,x-y)|}{|x-y|^{n-(\mu+m\beta)}}f(y)\mathrm{d}y \tag{1.8}$$

Applying the generalized Hölder's Inequality, we get

$$\int_{C_j}\left|\frac{|\Omega(x,x-y)|}{|x-y|^{n-(\mu+m\beta)}}f_j(y)\right|\mathrm{d}y\le\left\|f_j\right\|_{L^{p_1(\cdot)}}\left\|\frac{\Omega(x,x-y)}{|x-y|^{n-(\mu+m\beta)}}\chi_j\right\|_{L^{p_1'(\cdot)}}$$

Define the variable exponent $\frac{1}{p_1'(\cdot)} = \frac{1}{r} + \frac{1}{\widetilde{p_1'}(\cdot)}$ by Lemma 3.3, then we have

$$\begin{aligned}
&\int_{C_j} \left| \frac{\Omega(x, x-y)}{|x-y|^{n-(\mu+m\beta)}} f_j(y) \right| dy \\
&\le \|f_j\|_{L^{p_1(\cdot)}} \|\Omega(x, x-y)\|_{L^r} \left\| \frac{\chi_j}{|x-y|^{n-(\mu+m\beta)}} \right\|_{L^{\widetilde{p_1'}(\cdot)}} \\
&\le C2^{-j(n-(\mu+m\beta))} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} \left[\int_{2^{j-2}}^{2^j} r^{n-1} dr \left(\int_{S^{n-1}} |\Omega(x, y')|^r d\sigma(y') \right) \right]^{\frac{1}{r}} \\
&\le C2^{-j(n-(\mu+m\beta))} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} 2^{\frac{jn}{r}} \|\Omega\|_{L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})} \\
&\le C2^{-j(n-(\mu+m\beta))} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_j\|_{L^{\widetilde{p_1'}(\cdot)}} 2^{\frac{jn}{r}}
\end{aligned}$$

According Lemma 3.4 and the formula $\frac{1}{\widetilde{p_1'}(\cdot)} = \frac{1}{p_1'(\cdot)} - \frac{1}{r}$, we have $\|\chi_{B_j}\|_{L^{\widetilde{p_1'}(\cdot)}} \approx \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} |B_j|^{\frac{-1}{r}}$. Then we get

$$\begin{aligned}
\int_{C_j} \left| \frac{\Omega(x, x-y)}{|x-y|^{n-(\mu+m\beta)}} f_j(y) \right| dy &\le 2^{-j(n-(\mu+m\beta))} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} 2^{\frac{jn}{r}} 2^{\frac{-jn}{r}} \\
&\le 2^{-j(n-(\mu+m\beta))} \|f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}}
\end{aligned}$$

By (1.8), we can obtain that

$$\left| \left[b^m, T_{\Omega,\mu} \right] \right| \le C2^{-j(n-(\mu+m\beta))} \|b\|_{Lip_\beta}^m \|f_j\|_{L^{p_1(\cdot)}} \|\chi_{B_j}\|_{L^{p_1'(\cdot)}} \tag{1.9}$$

Then by (1.9) and Lemma 3.7, we have

$$\sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} \left[b^m, T_{\Omega,\mu} \right](f_j) \chi_k \right|}{\eta_1 \|b\|_{Lip_\beta}^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le C \sum_{k=-\infty}^{\infty} \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} \left[b^m, T_{\Omega,\mu} \right](f_j) \chi_k \right|}{\eta_1 \|b\|_{Lip_\beta}^m} \right)^{q_2(\cdot)} \right\|_{L^{p_2(\cdot)}}$$

where

$$(q_2^3)k = \begin{cases} (q_2)_-, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} \left[b^m, T_{\Omega,\mu} \right](f_j) \chi_k \right|}{\eta_1 \|b\|_{Lip_\beta}^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} \le 1 \\ (q_2)_+, & \left\| \left(\frac{2^{k\alpha} \left| \sum_{j=k+2}^{\infty} \left[b^m, T_{\Omega,\mu} \right](f_j) \chi_k \right|}{\eta_1 \|b\|_{Lip_\beta}^m} \right)^{q_2(\cdot)} \right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}} > 1 \end{cases}$$

Furthermore, when $\alpha > (\mu + m\beta) - n\delta_2$, note that $\|\chi_{B_k}\|_{L^{p_2(\cdot)}} \le C2^{-k(\mu+m\beta)} \|\chi_{B_k}\|_{L^{p_1(\cdot)}}$ see [9], the similar way to estimate η_{11}, we get

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}\left[b^{m},T_{\Omega,\mu}\right](f_j)\chi_k\right|}{\eta_1\|b\|_{Lip_\beta}^{m}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j(n-(\mu+m\beta))}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}\|\chi_{B_k}\|_{L^{p_2(\cdot)}}\right]^{(q_2^3)k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{-j(n-(\mu+m\beta))}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1'(\cdot)}}2^{-k(\mu+m\beta)}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^3)k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(j-k)(\mu+m\beta)}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\|\chi_{B_j}\|_{L^{p_1(\cdot)}}^{-1}\|\chi_{B_k}\|_{L^{p_1(\cdot)}}\right]^{(q_2^3)k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(j-k)(\mu+m\beta)}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\frac{\|\chi_{B_k}\|_{L^{p_1(\cdot)}}}{\|\chi_{B_j}\|_{L^{p_1(\cdot)}}}\right]^{(q_2^3)k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[2^{k\alpha}\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+m\beta))}\left\|\frac{f_j}{\eta_1}\right\|_{L^{p_1(\cdot)}(\mathbb{R}^n)}\right]^{(q_2^3)k}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+m\beta)+\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}^{\frac{1}{(q_1)_+}}\right]^{(q_2^3)k}$$

We can conclude that

$$\sum_{k=-\infty}^{\infty}\left\|\left(\frac{2^{k\alpha}\left|\sum_{j=k+2}^{\infty}\left[b^{m},T_{\Omega,\mu}\right](f_j)\chi_k\right|}{\eta_1\|b\|_{Lip_\beta}^{m}}\right)^{q_2(\cdot)}\right\|_{L^{\frac{p_2(\cdot)}{q_2(\cdot)}}}$$

$$\le C\sum_{k=-\infty}^{\infty}\left[\sum_{j=k+2}^{\infty}2^{(k-j)(n\delta_2-(\mu+m\beta)+\alpha)}\left\|\left(\frac{\left|2^{\alpha j}f_j\chi_k\right|}{\eta_1}\right)^{q_1(\cdot)}\right\|_{L^{p_1(\cdot)q_1(\cdot)}(\mathbb{R}^n)}\right]^{q_*}\le C$$

where $q_* = \min_{k\in\mathbb{N}}\frac{(q_2^3)k}{(q_1)_+}$, this implies that

$$\eta_{23}\le C\eta_1\|b\|_{Lip_\beta(\mathbb{R}^n)}^{m}\le C\|b\|_{Lip_\beta(\mathbb{R}^n)}^{m}\|f\|_{\dot{K}_{p_1(\cdot)}^{\alpha,q_1(\cdot)}(\mathbb{R}^n)}$$

This completes the proof Theorem 2.

Competing Interests

The authors declare that they have no competing interests.

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