Available online at www.sciencedirect.com
ScienceDirect
Transactions of A. Razmadze Mathematical Institute 171 (2017) 76–89
Transactions of A. Razmadze Mathematical Institute
www.elsevier.com/locate/trmi

Original article

Stochastic differential equations in a Banach space driven by the cylindrical Wiener process

Badri Mamporia

Niko Muskhelishvili Institute of computational Mathematics, Technical University of Georgia, 77 Kostava Str, Tbilisi, 0160, Georgia

Received 23 June 2016; received in revised form 17 October 2016; accepted 29 October 2016
Available online 21 November 2016

Abstract

Generalized stochastic integral from predictable operator-valued random process with respect to a cylindrical Wiener process in an arbitrary Banach space is defined. The question of existence of the stochastic integral in a Banach space is reduced to the problem of decomposability of the generalized random element. The sufficient condition of existence of the stochastic integral in terms of p-absolutely summing operators is given. The stochastic differential equation for generalized random processes is considered and existence and uniqueness of the solution is developed. As a consequence, the corresponding results of the stochastic differential equations in an arbitrary Banach space are given.

Keywords: Ito stochastic integrals and stochastic differential equations; Wiener processes; Covariance operators in Banach spaces

1. Introduction

First results on the infinite dimensional stochastic differential equations started to appear in the mid 1960s. The traditional finite dimensional methods gave desired results for Hilbert space case (see [1,2]), but they turned out deadlock in the general Banach space case. Then, researchers began to develop the problem in such Banach spaces, the geometry of which is close to the geometry of Hilbert space (see for example [3,4]). Important results are received in the case, when the Banach space has UMD property (see [5–7]). But the class of UMD Banach spaces is very narrow—they are reflexive Banach spaces. Stochastic analysis in UMD spaces intensively developed after the end of the eighties of the lust century, but the class of Banach spaces, where the traditional methods give desired results, has not yet extended. Numerous works are dedicated to this problem (see [8–10,6]). Therefore, it is greatly interesting to develop the stochastic differential equations in an arbitrary Banach space.

The first step to investigate this direction is to construct the Ito stochastic integral in an arbitrary separable Banach space. Stochastic integral for Banach space valued non random function by one dimensional Wiener process (the Wiener integral) is constructed in [11]. Stochastic integral from operator-valued non-random process by the Banach

E-mail address: badrimamporia@yahoo.com.
Peer review under responsibility of Journal Transactions of A. Razmadze Mathematical Institute.

http://dx.doi.org/10.1016/j.trmi.2016.10.003
2346-8092/

space valued Wiener process is considered in [12]. In [13] is constructed the stochastic integral from operator-valued (from Hilbert space to Banach space) non random function by the cylindrical Wiener process. There are also considered the traditional conditions of the existence of the stochastic integral with relation to the geometry of Banach space. The Ito stochastic integral in 2-uniformly smooth Banach spaces is considered in [3,14–16]. In [17] is shown, that the property of definition of 2-uniformly smooth Banach space is equivalent to the martingale type 2 property. Stochastic integral in UMD Banach spaces is constructed in [18,19,7]. In [20] is considered linear stochastic evolution equations on some special Banach spaces. We define the generalized stochastic integral in an arbitrary Separable Banach space for a wide class of non-anticipating operator-valued random processes by the cylindrical Wiener process, which is a generalized random element (a random linear function or a cylindrical random element), and if there exists the corresponding random element, that is, if this generalized random element is decomposable by the Banach space valued random element, then we say that this random element is the stochastic integral. Thus, the problem of existence of the stochastic integral in an arbitrary separable Banach space is reduced to the well known problem of decomposability of the generalized random element. We give the sufficient condition of existence of the stochastic integral using the L. Schwartz's and S. Kwapien's result in terms of p-absolutely summing operator (see [21,22]).

The second main problem to develop the stochastic differential equations in a Banach space is to estimate the stochastic integral, which is necessary for the iteration procedure to prove the existence and uniqueness of the solution. Such estimation is yet impossible in an arbitrary Banach space case. We consider the Banach space of generalized random elements and introduce there the stochastic differential equation for the generalized random process. For this situation, it is possible to use traditional methods to develop the problem of existence and uniqueness of the solution as a generalized random process. Afterward, from the main stochastic differential equation in an arbitrary Banach space we produce the equation for a generalized random process. As we have proved the existence and uniqueness of the solution of this equation, we receive the generalized random process as a solution of the produced stochastic differential equation. If this generalized random process is decomposable, then the corresponding Banach space valued random process will be the solution of the main stochastic differential equation in a Banach space. Therefore, we have also reduced the problem of existence of the solution of the stochastic differential equation in an arbitrary Banach space to the problem of decomposability of the generalized random element.

The investigation of the stochastic differential equations in a Banach space takes place in three directions. They can be described by means of the corresponding stochastic integrals in the equation. In the first (relatively) direction, the integrand non-anticipating process takes its values in a Banach space and the stochastic integral is taken by the scalar Wiener process. We considered this case in the paper [23]. In the second direction the integrand non-anticipating process is operator-valued (from Banach space to Banach space) and the stochastic integral is taken by the Wiener process in a Banach space. This case we investigated in the papers [24–26]. In the third direction the integrand is an operator-valued non-anticipating random function from Hilbert space to Banach space while the stochastic integral is taken by the cylindrical Wiener process in a Hilbert space. This article is devoted to this direction.

Now we give some definitions and preliminary results to realize our approach.

Let X be a real separable Banach space. X^*—its conjugate, $\mathcal{B}(X)$—the Borel σ-algebra of X, $(\Omega, \mathcal{B}, P)$—a probability space. The continuous linear operator $L : X^* \to L_2(\Omega, \mathcal{B}, P)$ is called a generalized random element (GRE). (Sometimes the terms: linear random function or cylindrical random element are used). We consider such GRE, which maps X^* to a fixed closed separable subspace $G \subset L_2(\Omega, \mathcal{B}, P)$. Denote $M_1 := L(X^*, G)$—the Banach space of GRE with the norm $\|L\| = \sup_{\|x^*\|\le 1} \|Lx^*\|_{L_2}$. A random element (measurable map) $\xi : \Omega \to X$ is said to have a weak second order, if, for all $x^* \in X^*$, $E\langle \xi, x^*\rangle^2 < \infty$. ξ we can realize as an element of $M_1 : L_\xi x^* = \langle \xi, x^*\rangle$. But in infinite dimensional spaces not every GRE may be represented by the Banach space valued random element. The problem of finding the conditions under which the GRE is represented by the Banach space valued random element is well known, otherwise also called the problem of decomposability of the GRE. This is the reason why we allot the superiority to the term GRE; GRE is a generalization of the random element in the infinite dimensional spaces. In the finite dimensional spaces every GRE is decomposable, thus, it is a random element. This term was early used by many authors (see for example [27,28,2,22] p. 140). Likewise, the problem of decomposability of the GRE is equal to the problem of extension of the finite additive (cylindrical) measure to the σ-additive measure. This is a reason why the term "cylindrical random element" appears.

Denote by M_2 the linear space of all random elements of the weak second order with the norm $\|\xi\| = \|L_\xi\|$. Therefore, we can assume $M_2 \subseteq M_1$.

Let $L \in M_1$. Consider the map $m_L : X^* \to R^1, m_L x^* = ELx^*$. m_L is linear and bounded, therefore $m_L \in X^{**}$, which is called the mean of the GRE L. When $L \in M_2$, that is, if there exists $\xi : \Omega \to X$ such that $Lx^* = \langle \xi, x^* \rangle$, then $m \in X$ (see [22] Th.2.3.1), and it is the Pettis integral of ξ. Further we consider the GRE with the mean 0.

The covariance operator of $L \in M_1$ is a symmetric and positive operator $R_L : X^* \to X^{**}, \langle R_L x^*, y^* \rangle = ELx^* Ly^*$ for all x^* and y^* from X^*. $R_L = L^* L$. It is known that if $L = L_\xi \in M_2$, then R_L maps X^* to X (see [22, Th.3.2.1]), and if R is a positive and symmetric linear operator from X^* to X, then there exist $(x_k^*)_{k \in N} \subset X^*$ and $(x_k)_{k \in N} \subset X$ such that $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k$, and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k, x^* \rangle x_k$ (see [22, Lemma 3.1.1]). In general, for a positive and symmetric linear operator $R_L : X^* \to X^{**}$ (as G is a separable subspace of $L_2(\Omega, \mathcal{B}, P)$), there exist $(x_k^*)_{k \in N} \subset X^*$ and $(x_k^{**})_{k \in N} \subset X^{**}$ such that $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k^{**}$, and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle x_k^{**}$.

Proposition 1. *Let T be a GRE. There exist $(x_k^*)_{k \in N} \subset X^*$ and $(x_k^{**})_{k \in N} \subset X^{**}$ such that for all $x^* \in X^*$, $Tx^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle Tx_k^*$, $ETx_k^* Tx_j^* = \langle R_T x_k^*, x_j^* \rangle = \delta_{kj}$, $R_T x_k^* = x_k^{**}$, $R_T x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle x_k^{**}$. Therefore, if T is a Gaussian, then Tx_k^*, $k = 1, 2, \ldots$ are independent, standard Gaussian random variables.*

Proof. Consider the covariance operator of the GRE T, $R_T : X^* \to X^{**}$, $R_T = T^* T$. Let $(x_k^*)_{k \in N} \subset X^*$ and $(x_k^{**})_{k \in N} \subset X^{**}$ be such that $\langle R_T x_k^*, x_j^* \rangle = \delta_{kj}$, $R_T x_k^* = x_k^{**}$, $R_T x^* = \sum_{k=1}^{\infty} \langle x^*, x_k^{**} \rangle x_k^{**}$, for all $x^* \in X^*$. If we take up $T_n x^* = \sum_{k=1}^{n} \langle x^*, x_k^{**} \rangle Tx_k^*$, then $E(Tx^* - T_n x^*)^2 = E(Tx^*)^2 - 2ETx^* T_n x^* + E(T_n x^*)^2 = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle^2 - 2 \sum_{k=1}^{n} \langle x_k^{**}, x^* \rangle^2 + \sum_{k=1}^{n} \langle x_k^{**}, x^* \rangle^2 = \sum_{k=n+1}^{\infty} \langle x_k^{**}, x^* \rangle^2 \to 0$.

Therefore $Tx^* = \sum_{k=1}^{\infty} \langle x^*, x_k \rangle Tx_k^*$.

If T is a Gaussian GRE, then Tx_k and Tx_m are independent for all $k \neq m$ as $ETx_k^* Tx_m^* = \langle R_T x_k^*, x_m^* \rangle = \delta_{k,m} = 0$. □

A family of GRE $(L_t)_{t \in [0,1]}$ is called a generalized random process (GRP). A weak second order Banach space valued random process $(\xi_t)_{t \in [0,1]}$ can be represented as a GRP: $L_{\xi_t} x^* = \langle \xi_t, x^* \rangle$. The GRP is called Gaussian, if for all $t_1, t_2, \ldots, t_n$ and $x_1^*, x_2^* \ldots, x_n$, the n-dimensional vector $(L_{t_1} x_1^*, L_{t_2} x_2^*, \ldots, L_{t_n} x_n^*)$ is a Gaussian vector in R^n.

Definition 1. The Gaussian generalized random process $(W_H(t))_{t \in [0,1]}$ in a separable Hilbert space H is called a cylindrical Wiener process, if for all h and g from H, and t, s, from $[0, 1]$, $EW_H(t)hW_H(s)g = \min(t, s)\langle h, g \rangle$.

Proposition 2. *Let $(W_H(t))_{t \in [0,1]}$ be a cylindrical Wiener process in H. For any orthonormal basis $(e_k)_{k \in N}$ in H, there exists the sequence of independent, standard, real valued Wiener processes $w_k(t)$ such that $W_H(t)h = \sum_{k=1}^{\infty} \langle e_k, h \rangle w_k(t)$.*

Proof. For any orthonormal basis $(e_k)_{k \in N}$ the random processes $W_H(t)e_k, k = 1, 2, \ldots$ are standard, one dimensional, independent Wiener processes in H. Therefore, $W_H(t)h = W_H(t) \sum_{k=1}^{\infty} \langle h, e_k \rangle e_k = \sum_{k=1}^{\infty} \langle h, e_k \rangle W_H(t)e_k = \sum_{k=1}^{\infty} \langle e_k, h \rangle w_k(t)$, where $w_k(t) \equiv W_H(t)e_k, k = 1, 2, \ldots$. □

Definition 2. The Gaussian GRP $(T_t)_{t \in [0,1]}$ is called a generalized Wiener process in a Banach space X, if, for all $x^* \in X^*$, $T_t x^*$ is one dimensional Wiener process and for all t, s from $[0, 1]$ and $y^* \in X^*$, $ET_t x^* T_s y^* = \min(t, s)\langle Rx^* y^* \rangle$, where $R : X^* \to X^{**}$ is the covariance operator of the GRE T_1.

Let R be the covariance operator of the GRE T_1, $R : X^* \to X^{**}$, by the factorization lemma (see [22, Lemma 3.1.1]) we have $R = A^* A$, where $A : H \to X^{**}$, H is a real separable Hilbert space.

Proposition 3. *Let $(T_t)_{t \in [0,1]}$ be a generalized Wiener process and R be the covariance operator of T_1, $R = AA^*$. $A : H \to X^{**}$. There exists the cylindrical Wiener process $(W_H(t))_{t \in [0,1]}$, in H such that $T_t = AW_H(t) = \sum_{k=1}^{\infty} Ae_k w_k(t)$, where $(e_k)_{k \in N}$ is an orthonormal basis in H and $w_k(t)$, $k = 1, 2, \ldots$ is a sequence of one dimensional independent Wiener processes. Therefore every generalized Wiener process in X is the "image" of the cylindrical Wiener process in a separable Hilbert space H.*

Proof. Let $R = AA^*$ be the covariance operator of the GRE T_1. We have $(x_k^*)_{k \in N} \subset X^*$ and $(x_k^{**})_{k \in N} \subset X^{**}$ such that, $\langle Rx_k^*, x_j^* \rangle = \delta_{kj}$, $Rx_k^* = x_k^{**}$ and for $x^* \in X^*$, $Rx^* = \sum_{k=1}^{\infty} \langle x_k^{**}, x^* \rangle x_k^{**}$. By the definition of the generalized Wiener process, $T_t x_k^*$, $k = 1, 2, \ldots$ are one dimensional Wiener processes, and for all t, s from $[0, 1]$ and x_j^*, $ET_t x_k^* T_s x_j^* = \min(t, s)\langle Rx_k^* x_j^* \rangle = \delta_{k,j}$. Therefore $T_t x_k^* := w_k(t)$, $k = 1, \ldots$ is a sequence of one dimensional independent Wiener processes.

Denote $T_n(t) = \sum_{k=1}^{n} Ae_k w_k(t) = \sum_{k=1}^{n} x_k^{**} T_t x_k^{*}$. Then, for any $x^* \in X^*$,

$$
\begin{aligned}
E(T_t x^* - T_n(t)x^*)^2 &= E(T_t x^*)^2 - 2ET_t x^* T_n(t)x^* + E(T_n(t)x^*)^2 \\
&= t\langle Rx^*, x^*\rangle - 2t\sum_{k=1}^{n}\langle Rx_k^*, x^*\rangle + t\sum_{k=1}^{n}\langle Rx_k^*, x_k^*\rangle \\
&= t\sum_{k=1}^{\infty}\langle x_k^{**}, x^*\rangle^2 - 2\sum_{k=1}^{n}\langle x_k^{**}, x^*\rangle^2 + \sum_{k=1}^{n}\langle x_k^{**}, x^*\rangle^2 = \sum_{k=n+1}^{\infty}\langle x_k^{**}, x^*\rangle^2 \to 0.
\end{aligned}
$$

That is, $T_t x^* = \lim T_n(t)x^* = \sum_{k=1}^{\infty}\langle Ae_k, x^*\rangle w_k(t) = \lim\langle A(\sum_{k=1}^{n} e_k w_k(t)), x^*\rangle = \langle AW_H(t), x^*\rangle$. □

Remark 1. In [29] we have analyzed the definition of the Wiener processes in a Banach space, where we have used the term "canonical generalized Wiener Process" instead of the term "cylindrical Wiener process". The term "cylindrical random element" appeared in relation to the cylindrical measures in vector spaces, as cylindrical random element (generalized random element) induces the finitely additive measure in a Banach space, which is naturally defined in the cylindrical algebra. We mentioned above the reason why we use the term GRE. In our opinion this term better responds to the purpose of the definition than the term "cylindrical random element". As the term "cylindrical Wiener process" is widely applied in literature, we also use this term here and intend to continue discussions on the terminology.

Remark 2. If $H = R^n$ and $(W_H(t))_{t\in[0,1]}$ is n-dimensional standard Wiener process $W_H(t) = (W_H(t)e_1, \dots, W_H(t)e_n) = (w_1(t), w_2(t), \dots, w_n(t))$, then, for all linear operators $A : R^n \to R^n$, $(AW_H(t))_{t\in[0,1]}$ is a Wiener process in R^n with covariance operator $R = AA^*$. For infinite dimensional H and bounded linear operator $A : H \to H$, $(AW_H(t))_{t\in[0,1]}$, $AW_H(t) = \sum_{k=1}^{\infty} Ae_k w_k(t)$ is a Hilbert space valued Wiener process with the covariance operator $R = AA^*$, if, and only if, A is a Hilbert–Schmidt operator. The generalized Wiener process in X, $(W_t)_{t\in[0,1]} \equiv (AW_H(t))_{t\in[0,1]}$, $A : H \to X$, is X-valued Wiener process, if, and only if, $R = AA^*$ is a Gaussian covariance. The sum $W_t = \sum_{k=1}^{\infty} Ae_k w_k(t)$ converges a.s. uniformly for t in X (see [30,31,25]).

Remark 3. Wiener process in a Banach space was first considered by L. Gross [32]. He introduced for it a special term—the measurable pseudonorm. The definition of the Wiener process introduced by L. Gross is unnatural in comparison with the definition of the finite dimensional Wiener process. The definition of the covariance operator of the Banach space valued random elements (see [33,22]) allows to consider Wiener process in a Banach space analogous to the finite dimensional case.

2. Stochastic integrals

2.1. Stochastic integral of the Hilbert space valued random function by the cylindrical Wiener process

Let $(W_H(t))_{t\in[0,1]}$ be a cylindrical Wiener process in H, $(F_t)_{t\in[0,1]}$—be the increasing family of σ-algebras such that (a) for all $h \in H$, $W_H(t)h$ is F_t-measurable for all $t \in [0, 1]$; (b) $W_H(s)h - W_H(t)h$ is independent to the σ-algebra F_t for all $s > t$. F_t contains all P-null sets from $\mathcal{B}$. We say that $(W_H(t))_{t\in[0,1]}$ is adapted to the family $(F)_{t\in[0,1]}$. Consider the non-anticipating function $\varphi : [0, 1] \times \Omega \to H$, that is, φ is $B([0, 1]) \times \mathcal{B}(\Omega)$-measurable and $\varphi(t)$ is F_t-measurable for all $t \in [0, 1]$.

We define the stochastic integral for a non-anticipating function $\varphi : [0, 1] \times \Omega \to H$, $\int_0^1\int_\Omega \|\varphi\|^2 dtdP < \infty$ by the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$.

If $\varphi(t, \omega)$ is a step function, $\varphi(t, \omega) = \sum_{k=0}^{n-1}\varphi(t_k)\chi_{[t_k,t_{k+1})}$, $0 = t_0 < t_1 < \cdots < t_n = 1$, $\varphi_{t_k} : \Omega \to H$, $k = 0, 1, \dots, (n - 1)$, then the stochastic integral of φ by the $(W_H(t))_{t\in[0,1]}$ is defined by the equality $\int_0^1 \varphi(t)dW_H(t) = \sum_{k=0}^{n-1}\langle W_H(t_{k+1}) - W_H(t_k), \varphi(t_k)\rangle$.

Let $(h_i)_{i\in N}$ be any orthonormal basis in H, then $W_H(t)h_i \equiv w_i(t)$ are independent F_t-adapted standard real valued Wiener processes and $\int_0^1 \varphi(t)dW_H(t) = \sum_{k=0}^{n-1}\sum_{i=1}^{\infty}\langle h_i, \varphi(t_k)\rangle(w_i(t_{k+1}) - w_i(t_k))$.

We have $E(\int_0^1 \varphi(t)dW_H(t))^2 = \sum_{k=0}^{n-1}(t_{k+1} - t_k)\sum_{i=1}^{\infty} E\langle\varphi_{t_k}, h_i\rangle^2 = \sum_{k=0}^{n-1} E\|\varphi(t_k)\|^2(t_{k+1} - t_k) = \int_0^1\int_\Omega \|\varphi(t, \omega)\|^2 dtdP$.

The following lemma will be used to define the stochastic integral of non-anticipating function from $L_2([0,1]\times\Omega, H)$.

Lemma 1. *For any non-anticipating function $\varphi(t,\omega)\in L_2([0,1]\times\Omega,H)$ there exists a sequence of non-anticipating step functions $\varphi_n(t,\omega)\in L_2([0,1]\times\Omega,H)$ such that $\varphi_n\to\varphi$ in $L_2([0,1]\times\Omega,H)$.*

Proof. Define $\phi_n(t,\omega)=\sum_{k=1}^{n}\langle\varphi(t,\omega),h_k\rangle h_k$. We have

$$\int_0^1 E\|\phi_n-\varphi\|^2dt=\int_0^1 E\left\|\sum_{k=1}^{n}\langle\varphi(t,\omega),h_k\rangle h_k-\sum_{k=1}^{\infty}\langle\varphi(t,\omega),h_k\rangle h_k\right\|^2dt$$

$$=\int_0^1 E\left\|\sum_{k=n+1}^{\infty}\langle\varphi(t,\omega),h_k\rangle h_k\right\|^2dt=\int_0^1 E\sum_{k=n+1}^{\infty}\langle\varphi(t,\omega),h_k\rangle^2dt\to 0.$$

For a fixed $k\in N$, let $(\varphi_{km})_{m\in N}$ be a sequence of real valued non-anticipating step functions such that $\varphi_{km}\to\langle\varphi,h_k\rangle$ in $L_2([0,1]\times\Omega)$, when $m\to\infty$. Let $\phi_{nm}=\sum_{k=1}^{n}\varphi_{km}h_k$. Then $\|\phi_{nm}-\phi_n\|_{L_2}^2=\sum_{k=1}^{n}\int_0^1\int_\Omega(\varphi_{km}-\langle\varphi,h_k\rangle)^2dtdP\ \to\ 0$. Therefore we can choose a subsequence $(\varphi_n)_{n\in N}$ of $((\phi)_{nm})_{n,m\in N}$ converging to φ in $L_2([0,1]\times\Omega,H)$. Lemma 1 is proved. □

Let $\varphi(t,\omega)\in L_2([0,1]\times\Omega,H)$ be a non-anticipating function. By Lemma 1, there exists the sequence of step functions $(\varphi_n)_{n\in N}$ converging to φ in $L_2([0,1]\times\Omega,H)$. Then as $E(\int_0^1\varphi_n(t)dW_H(t)-\varphi_m(t)dW_H(t))^2=\int_0^1 E\|\varphi_n-\varphi_m\|^2dt\to 0$, $n,m\to\infty$, we can define the stochastic integral for an arbitrary non-anticipating function $\varphi(t,\omega)\in L_2([0,1]\times\Omega,H)$.

Definition 3. Let $\varphi(t,\omega)\in L_2([0,1]\times\Omega,H)$ be a non-anticipating function. The limit of the sequence of the random variables $\int_0^1\varphi_n(t)dW_H(t)$ in $L_2(\Omega)$ is called the stochastic integral of φ by the cylindrical Wiener process in H, and is denoted by $\int_0^1\varphi(t)dW_H(t)$.

We can naturally define the stochastic integral $\int_0^t\varphi(s)dW_H(s)$ for all $t\in[0,1]$. It is easy to see that $\int_0^t\varphi(s)dW_H(s)=\sum_{k=1}^{\infty}\int_0^t\langle\varphi(s),e_k\rangle dw_k(s)$, where $(e_k)_{k\in N}$ is an arbitrary orthonormal basis in H and $(w_k(t)=W_H(t)e_k)_{t\in[0,1]}$, $k=1,2\ldots$ are independent one-dimensional standard Wiener processes.

2.2. *Stochastic integral of operator valued random process by the cylindrical Wiener process*

Let $(F)_{t\in[0,1]}$ be a filtration, $(\Omega,\mathcal{B},P)$, $(W_H(t))_{t\in[0,1]}$ be the cylindrical Wiener process in H adapted to $(F_t)_{t\in[0,1]}$, X be a real separable Banach space. Consider the Banach space of linear bounded operators $L(H,X)(L(X^*,H))$ from H to X (from X^* to H).

Definition 4. A function $\varphi(t,\omega):[0,1]\times\Omega\to L(H,X)$ is called non-anticipating with respect to $(F)_{t\in[0,1]}$, if

1. For all $h\in H$ the function $(t\times\omega)\to\varphi(t,\omega)h$ is measurable;
2. For all $h\in H$, $t\in[0,1]$ the function $\omega\to\varphi(t,\omega)h$ is F_t-measurable.

Definition 5. We say that a non-anticipating function $\varphi(t,\omega)\ :\ [0,1]\times\Omega\ \to\ L(H,X)$ belongs to the class $G(L(H,X))$ if

$$\sup_{\|x^*\|\le 1}\int_0^1\int_\Omega\|\varphi^*(t,\omega)x^*\|^2dtdP<\infty,$$

where $\varphi^*(t,\omega)$ is the conjugate of the operator $\varphi(t,\omega)$. We can define the norm in the linear space $G(L(H,X))$: $\|\varphi\|_G^2\equiv\sup_{\|x^*\|\le1}\int_0^1\int_\Omega\|\varphi^*(t,\omega)x^*\|^2dtdP$.

Let $\varphi\in G(L(H,X))$ and take any $x^*\in X^*$. φ^*x^* maps $[0,1]\times\Omega$ into H, $\int_0^1\int_\Omega\|\varphi^*x^*\|^2dtdP<\infty$ and it is non-anticipating. Therefore, we can define the stochastic integral $\int_0^1\varphi^*(t,\omega)x^*dW_H(t)$ which is a real random variable with variance $\int_0^1\int_\Omega\|\varphi^*(t,\omega)x^*\|^2dtdP$. Consider the map $T_\varphi:X^*\to L_2(\Omega,\mathcal{B},P)$, $T_\varphi x^*=\int_0^1\varphi^*(t,\omega)x^*dW_H(t)$. T_φ is a GRE.

Definition 6. Let $\varphi \in G(L(H, X))$. The generalized random element $T_\varphi : X^* \to L_2(\Omega, \mathcal{B}, P)$, $T_\varphi x^* = \int_0^1 \varphi^*(t,\omega) x^* dW_H(t)$ is called the generalized stochastic integral of the operator-valued random function φ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$.

Accordingly, we define the generalized stochastic integral $T_\varphi(t)x^* = \int_0^t \varphi^*(s,\omega)x^* dW_H(s)$, for all $t \in [0, 1]$.

We have $\int_0^t \varphi^*(s,\omega)x^* dW_H(s) = \sum_{k=1}^\infty \int_o^t \langle \varphi^*(s,\omega)x^*, e_k\rangle dw_k(t)$, where $w_k(t) = \langle W_H(t), e_k\rangle$, $k = 1, 2, \dots$ are one dimensional independent standard Wiener processes.

For any $\varphi \in G(L(H, X))$ the generalized stochastic integral as a GRE exists.

Let $\varphi \in G(L(H, X))$, $T_\varphi : X^* \to L_2(\Omega, \mathcal{B}, P)$ be a generalized stochastic integral of φ. Denote by $L_\varphi : X^* \to X^{**}$ the covariance operator of the GRE T_φ. It is easy to see that $L_\varphi = T_\varphi^* T_\varphi$.

Theorem 1. *The covariance operator of the generalized stochastic integral of an operator-valued random function $\varphi \in G(L(H, X))$ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$ has the form $L_\varphi x^* = \int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ and maps X^* to X (the double integral is meant in the sense of Pettis).*

Proof. Let us find the value of the operator L_φ on $x^* \in X^*$. For any $x_1^* \in X^*$, we have

$$\begin{aligned}\langle L_\varphi x^*, x_1^*\rangle &= ET_\varphi x^* T_\varphi x_1^* = E\int_0^1 \varphi(t,\omega)^* x^* dW_H(t) \int_0^1 \varphi(t,\omega)^* x_1^* dW_H(t)\\ &= \int_0^1\int_\Omega \langle \varphi^*(t,\omega)x^*, \varphi^*(t,\omega)x_1^*\rangle_H dt dP = \int_0^1\int_\Omega \langle \varphi\varphi^* x^*, x_1^*\rangle dt dP.\end{aligned}$$

Therefore the Pettis integral $\int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ as an element of X^{**} exists for all $x^* \in X^*$.

Let $(h_k)_{k\in N}$ be an orthonormal basis in H. Then

$$\begin{aligned}L_\varphi x^* &= \int_0^1\int_\Omega \varphi(t,\omega)\varphi^*(t,\omega)x^* dt dP = \int_0^1\int_\Omega \varphi(t,\omega)\left(\sum_{k=1}^\infty \langle \varphi^*(t,\omega)x^*, h_k\rangle h_k\right) dt dP\\ &= \int_0^1\int_\Omega \sum_{k=1}^\infty \langle \varphi(t,\omega)h_k, x^*\rangle \varphi(t,\omega)h_k dt dP.\end{aligned}$$

Denote $L_\varphi^{(n)} = \int_0^1 \int_\Omega \sum_{k=1}^n \langle \varphi(t,\omega)h_k, x^*\rangle \varphi(t,\omega)h_k dt dP$. Consider the random element $\varphi h_k : [0,1]\times\Omega \to X$, $k = 1, 2, \dots$. As φh_k is a random element of the weak second order, its covariance operator maps X^* to X and equals $L_k x^* = \int_0^1 \int_\Omega \langle \varphi(t,\omega)h_k, x^*\rangle \varphi(t,\omega)h_k dt dP$.

Therefore, for all n and x^*, $L_\varphi^{(n)} x^*$ belongs to X. As X is a closed subspace of X^{**}, it is enough to prove the convergence of the sequence $L_\varphi^{(n)} x^*$, $n = 1, 2, \dots$, to the $L_\varphi x^*$ in X^{**} for all $x^* \in X^*$. We have

$$\begin{aligned}\|L_\varphi x^* - L_\varphi^{(n)} x^*\| &= \sup_{\|x_1^*\|\le 1} \int_0^1\int_\Omega \sum_{k=n+1}^\infty \langle \varphi(t,\omega)h_k, x^*\rangle \langle \varphi(t,\omega)h_k, x_1^*\rangle dt dP\\ &\le \sup_{\|x_1^*\|\le 1}\left(\int_0^1\int_\Omega \sum_{k=n+1}^\infty \langle \varphi(t,\omega)h_k, x_1^*\rangle^2 dt dP\right)^{1/2}\\ &\quad\times\left(\int_0^1\int_\Omega \sum_{k=n+1}^\infty \langle \varphi(t,\omega)h_k, x^*\rangle^2 dt dP\right)^{1/2} \to 0.\end{aligned}$$

As we have

$$\left(\int_0^1\int_\Omega \sum_{k=n+1}^\infty \langle \varphi(t,\omega)h_k, x^*\rangle^2 dt dP\right)^{1/2} \to 0 \quad \text{and}$$

$$\sup_{\|x_1^*\|\le 1}\left(\int_0^1\int_\Omega \sum_{k=n+1}^\infty \langle \varphi(t,\omega)h_k, x_1^*\rangle^2 dt dP\right)^{1/2} \le \|L_\varphi x^*\| < \infty.$$

Therefore $L_\varphi^{(n)} x^* \to L_\varphi x^*$, $n \to \infty$. That is $L_\varphi x^* \in X$. Theorem 1 is proved. □

We defined the generalized stochastic integral for a wide class of non-anticipating operator-valued random functions $G(L(H, X))$. The generalized stochastic integral from $\varphi \in G(L(H, X))$ is GRE. This GRE T_φ is not always decomposable. That is, there does not always exist a random element $\xi : \Omega \to X$ such that $T_\varphi x^* = \langle \xi, x^* \rangle$, $x^* \in X^*$.

Definition 7. Let $\varphi \in G(L(H, X))$ be an operator-valued non-anticipating random function. We say that a random element $\xi : \Omega \to X$ (if such element exists) is the stochastic integral of φ with respect to a cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$ if for all $x^* \in X^* T_\varphi x^* = \langle \xi, x^* \rangle$ a.s. and write $\xi = \int_0^1 \varphi(t, \omega) dW_H(t)$.

Thus, the question of the existence of the stochastic integral is reduced to the problem of decomposability of the GRE. This problem is equivalent to the problem of extension of the weak second order cylindrical measure to the countable-additive measure. Therefore, to study the problem of the existence of the stochastic integral we can use the results in the mentioned fields.

Now we give a sufficient condition of existence of the stochastic integral from the operator-valued non-anticipating random process by the cylindrical Wiener process using the term of p-absolutely summing operators. In case of the Banach spaces the role of the Hilbert–Schmidt operator plays the p-absolutely summing operator.

Definition 8. A linear operator $A : H \to X$ is called p-absolutely summing, if there exist a constant $c > 0$ such that for all $n \in N$ and $h_1, h_2, \ldots, h_n$ from H

$$\left(\sum_{i=1}^{n} \|Ah_i\|^p\right)^{1/p} \le c \sup_{\|h\|\le 1} \left(\sum_{i=1}^{n} \langle h_i, h\rangle^p\right)^{1/p}.$$

If X is a Hilbert space, then for any $p \ge 1$ the class of the p-absolutely summing operators from H to H coincides with the class of the Hilbert–Schmidt operators (see [34, Corr. 3.16 and Th.4.10]].

By the factorization lemma, the covariance operator L_φ factorized through separable Hilbert space $L_\varphi = AA^*$, $A : H \to X$, if $(e_k)_{k\in N}$ is the orthonormal basis in H, then there exists $(x_k)_{k\in N}$ and $(x_k^*)_{k\in N}$ such that $Ae_k = x_k$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$ and $L_\varphi = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$.

Theorem 2. *Let $\varphi \in G(L(H, X))$ be an operator-valued non-anticipating random process, $L_\varphi x^* = \int_0^1 \int_\Omega \varphi\varphi^* x^* dt dP$ be the covariance operator of the generalized stochastic integral of φ with respect to the cylindrical Wiener process $(W_H(t))_{t\in[0,1]}$. If $L_\varphi = AA^*$ be such, that $A : H \to X$ is the p-absolutely summing operator for any $p \ge 2$, there exists the closed subspace $S \subset L_2(\Omega, B, P)$ such that for all $x^* \in X^* T_\varphi x^* \in S$ and $S \subset L_p(\Omega, B, P) \subset L_2(\Omega, B, P)$, then the stochastic integral $\xi = \int_0^1 \varphi(t, \omega) dW_H(t)$ exists, $E\|\xi\|^p < \infty$, $\xi = \sum_{k=1}^\infty x_k \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$ and the convergence is in $L_p(\Omega, X)$, where $Ae_k = x_k$, $\langle x_k, x_j^* \rangle = \delta_{k,j}$ and $L_\varphi = \sum_{k=1}^\infty \langle x_k, x^* \rangle x_k$.*

Proof. By Proposition 1, for any $x^* \in X^*$, we have $T_\varphi x^* = \int_0^1 \varphi^*(t, \omega) x^* dW_H(t) = \sum_{k=1}^\infty \langle x_k, x^* \rangle \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$. Since $T_\varphi x^* \in S$ and $S \subset L_p(\Omega, B, P)$, we can consider the identical map $I : S \to L_p(\Omega, B, P)$. By the closed graph theorem, I is a bounded operator, therefore, there exists $c > 0$, such that $(E(T_\varphi x^*)^p)^{\frac{1}{p}} \le c(E(T_\varphi x^*)^2)^{\frac{1}{2}}$. In a Hilbert space H consider the sum $\eta_n \equiv \sum_{k=1}^n e_k \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)$. For all $h \in H$, $\langle \eta_n, h \rangle$ converges in $L_p(\Omega, B, P)$ as

$$\begin{aligned}
(E(\langle \eta_n, h\rangle - \langle \eta_m, h\rangle)^p)^{\frac{1}{p}} &= \left(E\left(\sum_{k=n}^{m} \langle e_k, h\rangle \int_0^1 \varphi^*(t, \omega) x_k^* dW_H(t)\right)^p\right)^{\frac{1}{p}} \\
&= \left(E\left(\int_0^1 \varphi^*(t, \omega)\left(\sum_{k=n}^{m} \langle e_k, h\rangle x_k^*\right)\right)^p dt\right)^{\frac{1}{p}} dt \\
&\le c\left(E\left(\int_0^1 \varphi^*(t, \omega)\left(\sum_{k=n}^{m} \langle e_k, h\rangle x_k^*\right)\right)^2 dt\right)^{\frac{1}{2}} = \left(\sum_{k=n}^{m} \langle h, e_k\rangle^2\right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Here we used the following equalities: $\langle Lx_i^*, x_j^*\rangle = \delta_{i,j} = ET_\varphi x_i^* T_\varphi x_j^* = \int_0^1 \int_\Omega \langle \varphi^*(t,\omega)x_i^* \varphi^*(t,\omega)x_j^*\rangle dt dP$. That is, $(\eta_n)_{n\in N}$ is a sequence of the weak pth order random elements in H such that, for all $h \in H$, the sequence $\langle \eta_n, h\rangle$ converges in $L_p(\Omega, B, P)$. As A is a p-absolutely summing operator, by the lemma 6.5.2 of [22], the sequence $\sum_{k=1}^n Ae_k \int_0^1 \varphi^*(t,\omega)x_k^* dW_H(t)$ converges in $L_p(\Omega, X)$. Therefore, the stochastic integral $\xi = \int_0^1 \varphi(t,\omega)dW_H(t)$ exists, $\xi = \sum_{k=1}^\infty x_k \int_0^1 \varphi^*(t,\omega)x_k^* dW_H(t)$ and $E\|\xi\|^p < \infty$. □

Remark 4. Stochastic integral of operator-valued non-anticipating random process by the Wiener process in an arbitrary Banach spaces we considered in [24] (see also [25]), where we gave the sufficient condition of existence of the stochastic integral using p-absolutely summing operators.

Denote by $M_1^H := L(X^*, L_2(\Omega, B, P, H))$ the Banach space of generalized random elements with the norm $\|T\|^2 = \sup_{\|x^*\|\le 1} \int_\Omega \|Tx^*\|^2 dP < \infty$. If $\varphi : \Omega \to L(H, X)$ is such that for all $x^* \in X^*$, $\int_\Omega \|\varphi^* x^*\|^2 dP < \infty$, then, by the closed graph theorem, $T_\varphi : X^* \to L_2(\Omega, B, P, H)$, $T_\varphi x^* = \varphi^* x^*$ belongs to the space M_1^H. Denote by M_2^H the subspace of M_1^H of such GRE T_φ, that $\varphi : \Omega \to L(H, X)$ and $\int_\Omega \|\varphi^* x^*\|^2 dP < \infty$, for all $x^* \in X^*$. Consider now the family of linear bounded operators $(T_t)_{t\in[0,1]}$, $T_t : X^* \to L_2(\Omega, B, P, H)$ such that for all $x^* \in X^*$ the random process $T_t x^*$ is non-anticipating and $\sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|T_t x^*\|^2 dt dP < \infty$. Denote by TM_1^H the Banach space of such family of operators $(T_t)_{t\in[0,1]}$. We can define the generalized stochastic integral from $(T_t)_{t\in[0,1]} \in TM_1^H$.

Definition 9. Consider the GRP $(T_t)_{t\in[0,1]} \in TM_1^H$. The stochastic integral from $(T_t)_{t\in[0,1]}$ by the cylindrical Wiener process in H is the GRE defined by $I_T x^* = \int_0^1 T_t x^* dW_H(t)$, for all $x^* \in X^*$.

It is easy to see, that

$$I_T x^* = \int_0^1 T_t x^* dW_H(t) = \sum_{k=1}^\infty \int_0^1 \langle T_t x^*(\omega), e_k\rangle dw_k(t).$$

We have

$$\|I_T\|^2 = \sup_{\|x^*\|\le 1} E\left(\int_0^1 T_t x^* dW_H(t)\right)^2 = \sup_{\|x^*\|\le 1} \int_0^1 \int_\Omega \|T_t x^*\|^2 dt dP.$$

Accordingly, we have the isometrical operator

$$I : TM_1^H \to M_1, I((T_t)_{t\in[0,1]}) = \sum_{k=1}^\infty \int_0^1 \langle T_t x^*(\omega), e_k\rangle dw_k(t).$$

3. Stochastic differential equations

3.1. Stochastic differential equation for generalized random process driven by the cylindrical Wiener process

Consider now the Banach space of GRE M_1 and the stochastic differential equation for generalized random process in it:

$$dT_t = a(t, T_t)dt + B(t, T_t)dW_H(t), \tag{1}$$

with F_0-measurable initial condition $T_0 = L$, where $a : [0,1] \times M_1 \to M_1$ and $B : [0,1] \times M_1 \to M_1^H$.

Definition 10. A GRP $(T_t)_{t\in[0,1]}$ is called the strong generalized solution of Eq. (1) with the F_0-measurable initial condition $T_0 = L$, if the following assertions are true:

for all $x^* \in X^*$, $a(t, T_t)x^*$ and $B(t, T_t)x^*$ are $B[0,1] \times F_t$ measurable;

$E\int_0^1 (a(t, T_t)x^*)^2 dt + E\int_0^1 \|B(t, T_t)x^*\|^2 dt < \infty$; $T_t x^*$ is continuous, F_t-adapted random process and for all $t \in [0,1]$ and $x^* \in X^*$,

$T_t x^* = T_0 x^* + \int_0^t a(s, T_s)x^* ds + \int_0^1 B(s, T_s)x^* dW_H(s)$ a.s.

Definition 11. We say that the stochastic differential equation (1) has a unique strong generalized solution, if $(T_t)_{t\in[0,1]}$ and $(\overline{T_t})_{t\in[0,1]}$ are two solutions, then for each $x^* \in X^*$,

$$P(\omega : T_t(\omega)x^* = \overline{T_t}(\omega)x^* \quad \text{for all } t \in [0, 1]) = 1.$$

The following theorem gives the sufficient conditions of existence and uniqueness of a strong generalized solution to a stochastic differential equation for GRP.

Theorem 3. *Suppose that the coefficients of the stochastic differential equation (1) satisfy the following conditions:*

1. $\|a(t, T)\|^2_{M_1} + \|B(t, T)\|^2_{M_1^H} \le K^2(1 + \|T\|^2_{M_1})$,
2. $\|a(t, T) - a(t, S)\|^2_{M_1} + \|B(t, T) - B(t, S)\|^2_{M_1^H} \le K^2\|T - S\|^2_{M_1}$.

Then there exists a unique strong generalized solution $(T_t)_{t\in[0,1]}$ to (1) with initial condition $T_0 = L$, $L \in M_1$ and for all $x^* \in X^*$, Lx^* is F_0-measurable. The GRP $T : [0, 1] \to M_1$ is continuous.

Proof. To prove this Theorem we use the one dimensional technique which works here successfully. For all $t \in [0, 1]$, let $T_t^{(0)} = L$ and

$$T_t^{(n)}x^* = T_t^{(0)}x^* + \int_0^t a(s, T_s^{(n-1)})x^*ds + \int_0^t B^*(s, T_s^{(n-1)})x^*dW_H(s),$$

$$\begin{aligned}
\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} \le\ & 2 \sup_{\|x^*\|\le 1} E\left(\int_0^t (a(s, T_s^{(n)}) - a(s, T_s^{(n-1)}))x^*\, ds\right)^2 \\
& + 2 \sup_{\|x^*\|\le 1} \left(E\left(\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*dW_H(s)\right)\right)^2 \\
\le\ & 2\int_0^t \|a(s, T_s^{(n)}) - a(s, T_s^{(n-1)})\|^2_{M_1}ds + 2\int_0^t \|B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)})\|^2_{M_1^H}ds \\
\le\ & 2K^2 \int_0^t \|T_s^{(n)} - T_s^{(n-1)}\|^2_{M_1}ds.
\end{aligned}$$

Then we have

$$\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} \le (2K^2)^{(n-1)} \int_0^t \frac{(t-s)^{(n-1)}}{(n-1)!} \|T_s^{(1)} - T_s^{(0)}\|^2_{M_1}ds,$$

$$\|T_s^{(1)} - T_s^{(0)}\|^2_{M_1} \le 2\left\|\int_0^t a(s, T_s^{(0)})ds\right\|^2_{M_1} + 2\left\|\int_0^t (B^*s, T_s^{(0)})dW_H(s)\right\|^2_{M_1^H} \le 2K^2(1 + \|T_0\|^2_{M_1}).$$

Consequently, $\|T_t^{(n+1)} - T_t^{(n)}\|^2_{M_1} \le pC^n/n!$ for any positive p and C.

It is easy to see, that $\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)})x^*)dW_H(s)$ is a martingale for all fixed $x^* \in X^*$ and therefore,

$$\begin{aligned}
& E \sup_{0\le t\le 1} \left|\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\, dW_H(s)\right|^2 \le 4\int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\|^2ds \\
& \le 4 \sup_{\|x^*\|\le 1} \int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\|^2ds \\
& \le 4\int_0^1 E\|(B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))\|^2_{M_1^H}ds.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
& E \sup_{0\le t\le 1} |(T_t^{(n+1)} - T_t^{(n)})x^*|^2 \le 2E \sup_{0\le t\le 1} \int_0^t ((a(s, T_s^{(n)}) - a(s, T_s^{(n-1)}))x^*)^2ds \\
& \quad + 2E \sup_{0\le t\le 1} \left|\int_0^t (B^*(s, T_s^{(n)}) - B^*(s, T_s^{(n-1)}))x^*\, dW_H(s)\right|^2
\end{aligned}$$

$$\leq 2\int_0^1 \|a(s,T_s^{(n)}) - a(s,T_s^{(n-1)})\|_{M_1}^2 ds + 8\int_0^1 \|B^*(s,T_s^{(n)}) - B^*(s,T_s^{(n-1)})\|_{M_1}^2 ds$$
$$\leq \frac{10pC^{n-1}}{(n-1)!}.$$

Then we have

$$\sum_{n=1}^{\infty} P\Big(\sup_{0\leq t\leq 1} |(T_t^{(n+1)} - T_t^{(n)})x^*| \geq \frac{1}{n^2}\Big) \leq \sum_{n=1}^{\infty} n^4 E\left(\sup_{0\leq t\leq 1} |(T_t^{(n+1)} - T_t^{(n)})x^*|^2 \leq 10p\sum_{n=1}^{\infty} \frac{n^4C^{n-1}}{(n-1)!}\right).$$

By the Borel–Cantelli lemma, the series $T_t^{(0)}x^*(\omega) + \sum_{n=1}^{\infty}(T_t^{(n)}(\omega) - T_t^{(n-1)}(\omega))x^*$ converges uniformly for t (P-a.s.) to the continuous random process which we denote by T_tx^*, $x^* \in X^*$. Therefore, we get GRP $T_t : X^* \to L_2(\Omega, B, P)$. From Eq. (2) we obtain

$$T_tx^* = T^{(0)}x^* + \int_0^t a(s,T_s)x^*ds + \int_0^t B^*(s,T_s)x^*dW_H(s) \quad \text{a.s.}$$

Therefore, the GRP (T_t), $t \in [0,1]$, constructed above, is a strong generalized solution of Eq. (1).

The uniqueness of the solution we can prove similarly to the finite dimensional case. □

3.2. Stochastic differential equation in an arbitrary Banach space driven by the cylindrical Wiener process

Let us now consider the stochastic differential equation in an arbitrary Banach space

$$d\xi_t = a(t,\xi_t)dt + B(t,\xi_t)dW_H(t), \tag{2}$$

where $a : [0,1]\times X \to X$ and $B : [0,1]\times X \to L(H,X)$ are such functions that $a(t,\xi) \in M_2$ and $B^*(t,\xi) \in M_2^H$ for all $t \in [0,1]$ and for all weak second order random elements ξ; and the following inequalities hold at that:

1′. $\|a(t,\xi)\|_{M_1}^2 + \|B^*(t,\xi)\|_{M_1^H}^2 \leq K^2(1 + \|\xi\|_{M_1}^2)$,

2′. $\|a(t,\xi) - a(t,\eta)\|_{M_1}^2 + \|B^*(t,\xi) - B^*(t,\eta)\|_{M_1^H}^2 \leq K^2\|\xi - \eta\|_{M_1}^2$, where ξ and η are weak second order X-valued random elements.

We can extend the coefficients a and B on $\overline{M_2} \subset M_1$ correspondingly: Let $T \in \overline{M_2}$, there exists $(\xi_n)_{n\in N} \subset M_2$ such that $\|\xi_n - T\|_{M_1} \to 0$. Then $\|a(t,\xi_n) - a(t,\xi_m)\|_{M_1}^2 \leq K^2\|\xi_n - \xi_m\|_{M_1}^2 \to 0$ and $\|B(t,\xi_n)h - B(t,\xi_m)h\|_{M_1}^2 \leq K^2\|h\|^2\|\xi_n - \xi_m\|_{M_1}^2 \to 0$. $\|B^*(t,\xi_n) - B^*(t,\xi_m)\|_{M_1^H}^2 \leq K^2\|\xi_n - \xi_m\|_{M_1}^2 \to 0$. Denote $a(t,T) := \lim_{n\to\infty} a(t,\xi_n)$, $B(t,T)h := \lim_{n\to\infty} B(t,\xi_n)h$ and $B^*(t,T) := \lim B^*(t,\xi_n)$. $a(t,T) \in \overline{M_2}$, $B(t,T)h \in \overline{M_2}$ and $B^*(t,T) \in \overline{M_2^H} \subset M_1^H$. Therefore, we receive from Eq. (2) the corresponding stochastic differential equation for GRP:

$$dT_t = a(t,T_t)dt + B^*(t,T_t)dW_H(t), \tag{3}$$

with initial condition $T_0x^* = \langle\xi_0, x^*\rangle$. It is easy to see that the coefficients of this equation satisfy the conditions 1 and 2 of Theorem 2.

Remember that we have the condition $B^*(t,\xi) \in M_1^H$, that is $\sup_{\|x^*\|\leq 1} E\|B^*x^*\|^2 = \sup_{\|x^*\|\leq 1}\sum_{k=1}^{\infty} E\langle B^*(t,\xi)x^*, e_k\rangle^2 < \infty$. Further we need the following assertion:

$$\sup_{\|x^*\|\leq 1}\sum_{k=n}^{\infty} E\langle B^*(t,\xi)x^*, e_k\rangle^2 \to 0. \tag{4}$$

It is easy to see, that if $B^*(t,\xi)$ satisfies the condition (4) for all $\xi \in M_2$, then this condition is true for all $T \in \overline{M_2}$.

Then we have the following theorem:

Theorem 4. *If the coefficients of Eq. (2) satisfy the conditions* 1′, 2′, (4) *and for all* $\xi \in M_2$, $a(.,\xi)$ *from* $[0,1]$ *to* $\overline{M_2}$ *and* $B^*(.,\xi)$ *from* $[0,1]$ *to* $\overline{M_2^H}$ *are continuous, then the corresponding stochastic differential equation* (3) *possesses a unique strong generalized solution with initial condition* $T_0x^* = \langle\xi_0, x^*\rangle$. *The solution* $(T_t)_{t\in[0,1]}$ *is such that* $T_t \in \overline{M_2}$ *for all* $t \in [0,1]$.

Proof. To use Theorem 2, it is enough to prove that in the iteration formula

$$T_t^{(n)}x^* = T_t^{(0)}x^* + \int_0^t a(s, T_s^{(n-1)})x^* ds + \int_0^t B^*(s, T_s^{(n-1)})x^* dW_H(s), \tag{5}$$

the members $\int_0^t a(s, T_s^{(n-1)})x^* ds$ and $\int_0^t B^*(s, T_s^{(n-1)})x^* dW_H(s)$ of the formula (5) belong to the space $\overline{M_2}$, where $T^{(0)}x^* = \langle \xi_0, x^* \rangle$. As we showed above, $a(t, T)$ and $B(t, T)h$ belong to $\overline{M_2}$ for all $h \in H$.

In [23] we defined the generalized stochastic integral from the non-anticipating weak second order Banach space valued random processes (from the non-anticipating GRP) by one dimensional standard Wiener process. If $\varphi(t, \omega) \in G(L(H, X))$ is a non-anticipating function, $(W_H(t))_{t\in[0,1]}$, $W_H(t) = \sum_{k=1}^{\infty} e_k w_k(t)$ is the cylindrical Wiener process for any $(e_k)_{k\in N}$ orthonormal basis of H, then $\varphi(t, \omega)e_k$ is X-valued non-anticipating random process for all $k \in N$. The generalized stochastic integral $\int_0^t \varphi(t, \omega)e_k dw_k(t)$ exists. This stochastic integral belongs to $\overline{M_2}$; moreover, if $L : [0, 1] \to \overline{M_2}$, is continuous, $\int_0^1 \|L(t)\|_{M_1}^2 < \infty$, then $\int_0^1 L(t)dw_t \in \overline{M_2}$ (see [23], Theorem 2). We will use this result to prove that $I(t) : X^* \to L_2(\Omega, B, P)$, $I(t)x^* = \int_0^t B^*(s, T_s^{(n-1)})x^* dW_H(s)$ belongs to $\overline{M_2}$ for all $n \in N$: $x^* \to \int_0^t \langle B^*(s, T_s^0)x^*, e_k \rangle dw_k(s)$ belongs to $\overline{M_2}$. If $T_s^{(n-1)}$ belongs to $\overline{M_2}$, then $x^* \to \int_0^t \langle B^*(s, T_s^{(n-1)})x^*, e_k \rangle dw_k(s)$ belongs to $\overline{M_2}$. Therefore, $I_m(t) : X^* \to L_2(\Omega, B, P)$, $I_m(t)x^* := \sum_{k=1}^{m} \int_0^t \langle B^*(s, T_s^{(n-1)})x^*, e_k \rangle dw_k(s)$ belongs to $\overline{M_2}$; $\|I(t) - I_m(t)\|_{M_1}^2 = \| \sum_{k=m+1}^{\infty} \int_0^t B(s, T_s^{(n-1)})e_k dw_k(s)\|_{M_1}^2 = \sup_{\|x^*\|\le 1} \sum_{k=m+1}^{\infty} \int_0^t E\langle B^*(t, T^{(n-1)})x^*, e_k \rangle^2 \to 0$ by the condition (4) and Lebesgue Theorem, as $\sup_{\|x^*\|\le 1} \sum_{k=m+1}^{\infty} E\langle B^*(t, T^{(n-1)})x^*, e_k \rangle^2 \le \sup_{\|x^*\|\le 1} \sum_{k=1}^{\infty} E\langle B^*(t, T^{(n-1)})x^*, e_k \rangle^2 = \sup_{\|x^*\|\le 1} E\|B^* x^*\|^2 = \|B^*(t, T^{(n-1)})x^*\|_{M_1^H}^2 \le K^2(1 + \|T^{(n-1)}\|^2) < \infty$, we have $I(t) \in \overline{M_2}$. □

Consequently, we receive the GRP $(T_t)_{t\in[0,1]} \in \overline{M_2}$,

$$T_t x^* = \langle \xi_0, x^* \rangle + \int_0^t \langle a(s, T_s), x^* \rangle ds + \int_0^t B^*(s, T_s)x^* dW_H(s) \tag{6}$$

as a generalized solution of the stochastic differential equation (3) corresponding to the stochastic differential equation (2) in an arbitrary separable Banach space.

Consider now the members of the equality (6): denote $T_t' x^* = \int_0^t \langle a(s, T_s), x^* \rangle ds + \int_0^t B^*(s, T_s)x^* dW_H(s)$. Let L_1' be the covariance operator of the GRE T_1'. By Theorem 1, the operator L_1' maps X^* to X. Let $L_1' = A'A'^*$ be the factorization of the covariance operator L_1', $A' : H \to X$. From Theorems 2 and 4 we receive the following:

Corollary 1. *If the GRE T_1' satisfies the conditions of Theorem 2, in particular, if the operator $A' : H \to X$ is 2-absolutely summing, then there exists the X-valued random process $(\xi_t)_{t\in[0,1]}$ such that $E\|\xi_t\|^2 < \infty$ and $\xi_t = \xi_0 + \int_0^t a(s, \xi_s)ds + \int_0^t B(s, \xi_s)dW_H(s)$, that is, $(\xi_t)_{t\in[0,1]}$ is the solution of the stochastic differential equation (2) in an arbitrary separable Banach space.*

Consider now a linear stochastic differential equation in a separable Banach space.

$$d\xi_t = A(t)\xi_t dt + B(t)\xi_t dW_H(t), \tag{7}$$

where $A : [0, 1] \to L(X, X)$ and $B : [0, 1] \to L(X, L(H, X))$ are continuous and $B(t, x)$ is such, that there exists $(e_k)_{k\in N}$ the orthonormal basis in H with the property $\sup_{t\in[0,1]} \sup_{\|x^*\|\le 1} \sum_{k=n}^{\infty} \|B(t)^* \delta_{e_k, x^*}\|^2 \to 0$, where δ_{e_k, x^*} is an element of $L(H, X)^*$, $\langle C, \delta_{e_k, x^*} \rangle = \langle Ce_k, x^* \rangle$, for all $C \in L(H, X)$. Denote $D \equiv \sup_{t\in[0,1]} \sup_{\|x^*\|\le 1} \sum_{k=1}^{\infty} \|B(t)^* \delta_{e_k, x^*}\|^2$. Then $\max_{t\in[0,1]}(\|A(t)\|, D) \equiv M < \infty$. For all weak second order random elements ξ, we have

$$\begin{aligned}\|A(t)\xi\|_{M_1}^2 &= \sup_{\|x^*\|\le 1} E\langle A(t)\xi, x^* \rangle^2 = \sup_{\|x^*\|\le 1} E\langle \xi, A^*(t)x^* \rangle^2 \\ &= \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\left\langle \xi, \frac{A^*(t)x^*}{\|A^*(t)\|} \right\rangle^2 \le \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\langle \xi, x^* \rangle^2 \le M^2(1 + \|\xi\|_{M_1}^2),\end{aligned}$$

and for all weak second order random elements ξ and η we have also

$$\begin{aligned}\|A(t)\xi - A(t)\eta\|^2_{M_1} &= \sup_{\|x^*\|\le 1} E\langle A(t)(\xi-\eta), x^*\rangle^2 \\ &= \sup_{\|x^*\|\le 1} E\langle(\xi-\eta), A^*(t)x^*\rangle^2 = \|A^*(t)\|^2 \sup_{\|x^*\|\le 1} E\left\langle(\xi-\eta), \frac{A^*(t)x^*}{\|A^*(t)\|}\right\rangle^2 \\ &\le M^2 \sup_{\|x^*\|\le 1} E\langle(\xi-\eta), x^*\rangle^2 = M^2\|\xi-\eta\|^2_{M_1}.\end{aligned}$$

Further, for all weak second order random elements ξ

$$\begin{aligned}\|B(t)\xi\|^2_{M_1^H} &= \sup_{\|x^*\|\le 1} E\|(B(t)\xi)^*x^*\|^2 \\ &= \sup_{\|x^*\|\le 1} E\sum_{k=1}^{\infty}\langle(B(t)\xi)^*x^*, e_k\rangle^2 = \sup_{\|x^*\|\le 1} E\sum_{k=1}^{\infty}\langle\xi, B(t)^*\delta_{x^*,e_k}\rangle^2 \\ &\le \sup_{\|x^*\|\le 1}\sum_{k=1}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 E\left\langle\xi, \frac{B(t)^*\delta_{x^*,e_k}}{\|B(t)^*\delta_{e_k,x^*}\|}\right\rangle^2 \\ &\le \sup_{\|x^*\|\le 1}\sum_{k=1}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \sup_{\|x^*\|\le 1} E\langle\xi, x^*\rangle^2 \le M^2\|\xi\|^2_{M_1}.\end{aligned}$$

Analogously, we can receive the inequality $\|B(t)\xi - B(t)\eta\|^2_{M_1^H} \le M^2\|\xi-\eta\|^2_{M_1}$.

Further,

$$\begin{aligned}\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty} E\langle B^*(t,\xi)x^*, e_k\rangle^2 &= \sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty} E\langle(B(t)\xi)^*x^*, e_k\rangle^2 \\ &= \sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{x^*e_k}\|^2 E\left\langle\xi, \frac{B(t)^*\delta_{x^*,e_k}}{\|B(t)^*\delta_{x^*e_k}\|}\right\rangle^2 \\ &\le \|\xi\|^2_{M_1}\cdot\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{x^*e_k}\|^2 \to 0.\end{aligned}$$

Therefore, if there exists $(e_k)_{k\in N}$ the orthonormal basis in H, such that $\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \to 0$, then for the linear stochastic differential equation (7) the conditions $1'$ and $2'$ and (4) are satisfied. Thus, by Theorem 4 we have the following:

Theorem 5. *For the linear stochastic differential equation (7), if there exists $(e_k)_{k\in N}$ the orthonormal basis in H with the property $\sup_{t\in[0,1]}\sup_{\|x^*\|\le 1}\sum_{k=n}^{\infty}\|B(t)^*\delta_{e_k,x^*}\|^2 \to 0$, then there exists the unique generalized solution of this equation $(T_t)_{t\in[0,1]}$, $T_t \in \overline{M_2}$ for all $t \in [0,1]$ with the initial condition $T_0x^* = \langle\xi_0, x^*\rangle$, where $\xi_0 \in M_2$ is F_0-measurable.*

In [26] we considered the stochastic differential equation driven by the Wiener process in a Banach space. If $R = UU^*$ is a Gaussian covariance in a Banach space, then $W_t \equiv UW_H(t) = \sum_{k=1}^{\infty} Ue_kw_k(t)$, $t \in [0,1]$ is a Wiener process in a Banach space for all orthonormal bases in H and convergence we have in $C([0,1], X)$. If $A : [0,1] \to L(X,X)$ and $B : [0,1] \to L(X, L(X,X))$ are continuous, then by Theorem 2 of [26], we have the following.

Corollary 2. *For the linear stochastic differential equation $d\xi_t = A(t)\xi_t dt + (B(t)\xi_t)UdW_H(t)$, where $A : [0,1] \to L(X,X)$ and $B : [0,1] \to L(X, L(X,X))$ are continuous and $R = UU^*$ is a Gaussian covariance, there exists the unique generalized solution $(T_t)_{t\in[0,1]}$, $T_t \in \overline{M_2}$ for all $t \in [0,1]$ with the initial condition $T_0x^* = \langle\xi_0, x^*\rangle$, where $\xi \in M_2$ is F_0-measurable.*

Acknowledgments

The author would like to thank reviewers for helpful comments.

References

[1] T. Chantladze, A stochastic differential equation in Hilbert space, Soobsc. Akad. Nauk Gruzin. SSR 33 (1964) 529–534.
[2] Ju.L. Daleckiĭ, Differential equations with functional derivatives and stochastic equations for generalized random processes, Dokl. Akad. Nauk SSSR 166 (1966) 1035–1038 (in Russian).
[3] Ja.I. Belopol'skaja, Ju.L. Dalec'kiĭ, Diffusion processes in smooth Banach spaces and manifolds. I, Tr. Mosk. Mat. Obs. 37 (1978) 107–141, 270 (in Russian).
[4] K. Itô, Foundations of Stochastic Differential Equations in Infinite-Dimensional Spaces, SIAM, Philadelphia, Pennsylvania, 1984; L. Gross, Potential theory on Hilbert space, J. Funct. Anal. 1 (1967) 123–181.
[5] Z. Brzeźniak, J.M.A.M. van Neerven, M.C. Veraar, L. Weis, Itô's formula in UMD Banach spaces and regularity of solutions of the Zakai equation, J. Differential Equations 245 (1) (2008) 30–58.
[6] J.M.A.M. Van Neerven, Stochastic Evolution Equations, in: ISEM Lecture Notes, 2007–2008.
[7] J.M.A.M. Van Neerven, M.C. Veraar, L. Weis, Stochastic integration in UMD Banach spaces, Ann. Probab. 35 (4) (2007) 1438–1478.
[8] G. Da Prato, J. Zabczyk, Stochastic Equations in Infinite Dimensions, in: Encyclopedia of Mathematics and its Applications, vol. 44, Cambridge University Press, Cambridge, 1992.
[9] L. Gavarecki, V. Mandrekar, Stochastic Differential Equations in Infinite Dimensions with Applications to Stochastic Partial Differential Equations, in: Probability and its Applications (New York), Springer, Heidelberg, 2011.
[10] G. Kallianpur, Xiong Jie, Stochastic Differential Equations in Infinite-Dimensional Spaces, in: Lecture Notes-Monograph Series, vol. 26, 1995.
[11] J. Rosinski, Z. Suchanecki, On the space of vector-valued functions integrable with respect to the white noise, Colloq. Math. 43 (1) (1980) 183–201. (1981).
[12] B. Mamporia, Stochastic integral from operator-valued function by the Wiener process in a Banach space, Soobsc. Akad. Nauk Gruzin. SSR 105 (1) (1982) 29–32 (in Russian).
[13] J.M.A.M. Van Neerven, L. Weis, Stochastic integration of functions with values in a Banach space, Studia Math. 166 (2) (2005) 131–170.
[14] Z. Brzeźniak, Some remarks on Itô and Stratonovich integration in 2-smooth Banach spaces, in: Probabilistic Methods in Fluids, World Sci. Publ., River Edge, NJ, 2003, pp. 48–69.
[15] E. Dettweiler, Stochastic integration relative to Brownian motion on a general Banach space, Doğa Mat. 15 (2) (1991) 58–97.
[16] A.L. Neidhardt, Stochastic Integrals in 2-Uniformly Smooth Banach Spaces (Ph. D. thesis), University of Wisconsin, 1978.
[17] G. Pisier, Martingales with values in uniformly convex spaces, Israel J. Math. 20 (3–4) (1975) 326–350.
[18] D.J.H. Garling, Brownian motion and UMD-spaces, in: Probability and Banach Spaces (Zaragoza, 1985), in: Lecture Notes in Math., vol. 1221, Springer, Berlin, 1986, pp. 36–49.
[19] T.R. McConnell, Decoupling and stochastic integration in UMD Banach spaces, Probab. Math. Statist. 10 (92) (1989) 283–295.
[20] Z. Brzeźniak, Honglwel Long, A note on γ-Radonifying and summing operators, in: Stochastic Analysis Banach Center Publications, Vol. 105, Polish Acad. Sci. Inst. of Math. Warszawa, 2015.
[21] W. Linde, V.I. Tarieladze, S.A. Chobanyan, A probabilistic characterization of summing operators, Mat. Zametki 30 (1) (1981) 133–142, 155 (in Russian).
[22] N.N. Vakhania, V.I. Tarieladze, S.A. Chobanyan, Probability Distributions on Banach Spaces, in: Mathematics and its Applications (Soviet Series), vol. 14, D. Reidel Publishing Co., Dordrecht, 1987, Translated from the Russian and with a preface by Wojbor A. Woyczynski.
[23] B. Mamporia, A stochastic differential equation for generalized random processes in a Banach space, Teor. Veroyatn. Primen. 56 (4) (2011) 704–725. translation in Theory Probab. Appl. 56 (4) (2012), 602–620 (in Russian).
[24] B. Mamporia, Stochastic integral from operator-valued random process by the Wiener process in a Banach space, Soobsc. Akad. Nauk Gruzin. SSR 105 (3) (1982) 17–21 (in Russian).
[25] B. Mamporia, Wiener processes and stochastic integrals in a Banach space, Probab. Math. Statist. 7 (1) (1986) 59–75.
[26] B. Mamporia, Stochastic differential equation driven by the Wiener process in a Banach space, existence and uniqueness of the generalized solution, Pure Appl. Math. J. 4 (3) (2015) 133–138. View 2 06, Downloads 26, doi: 11648/j.pamj.20150403.22.
[27] S.A. Chobanyan, Some characterizations of Gaussian Measures in a Banach spaces, in: Proc. of Computing Center of Academy of Science of Georgian Soviet Republic, Tbilisi, 13, 14, 2, 1975 (in Russian).
[28] S.A. Chobanian, A. Weron, Banach-Space-Valued Stationary Processes and their Linear Prediction, Institute Matematiezny Polskiej Academy Nauk, Warszawa, 1975, http://eudml.org/doc/268443.
[29] B. Mamporia, On the Wiener processes in a Banach space, Bull. Georgian Natl. Acad. Sci. (NS) 7 (2) (2013) 5–14.
[30] S. Chevet, Seminaire sur La Geometre des Espaces de Banach. Ecole Politechnique, Centre de Mathematicue, Exp. 19, 1977–1978.
[31] B. Mamporia, On Wiener processes in a Frechet space, Soobsc. Akad. Nauk Gruzin. SSR 87 (3) (1977) 549–552 (in Russian).
[32] L. Gross, Potential theory on Hilbert space, J. Funct. Anal. 1 (1967) 123–181.
[33] N.N. Vakhania, Probability Distributions on Linear Spaces, in: North-Holland Series in Probability and Applied Mathematics, North-Holland Publishing Co., New York-Amsterdam, 1981, Translated from the Russian by I.I. Kotlarski.
[34] J. Diestel, H. Jarchow, A. Tonge, Absolutely Summing Operators, in: Cambridge Studies in Advanced Mathematics, vol. 43, Cambridge University Press, Cambridge, 1995.

Further reading

[1] Byoung Jin Choi, Jin Pil Choi, Ji Un Cig, Stochastic Calculus for Banach space valued regular stochastic Processes, J. Chungcheng Math. Soc. 24 (1) (2011) 45–57.

[2] J. Dettweiler, L. Weis, Van Neerven, Space-time regularity of solutions of the parabolic stochastic Cauchy problem, Stoch. Anal. Appl. 24 (4) (2006) 843–869.

[3] V. Mandrekar, B. Rüdiger, Stochastic Integration in Banach Spaces. Theory and Applications, in: Probability Theory and Stochastic Modelling, vol. 73, Springer, Cham, 2015.

[4] B.L. Rozovskiĭ, Stochastic Evolution Systems. Linear Theory and Applications to Nonlinear Filtering, in: Mathematics and its Applications (Soviet Series), vol. 35, Kluwer Academic Publishers Group, Dordrecht, 1990, Translated from the Russian by A. Yarkho.

[5] N.N. Vakhania, N.P. Kandelaki, A stochastic integral for operator-valued functions, Theory Probab. Appl. 12 (3) (1967) 525–528.

[6] J.B. Walsh, An introduction to stochastic partial differential equations. École d'Été de Probabilités de Saint Flour XIV-1984, pp. 265–439.