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Short Communication

Simplifying differential equations concerning degenerate Bernoulli and Euler numbers

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Abstract

In the paper, the authors significantly and meaningfully simplify two families of nonlinear ordinary differential equations in terms of the Stirling numbers of the first and second kinds.

Keywords: Nonlinear ordinary differential equation; Stirling number of the first kind; Stirling number of the second kind; Bernoulli number; Euler number

1. Motivations and main results

In this section, we state three motivations and our main results of current paper.

1.1. First motivation

In [1, Theorems 1 and 2], the authors spent five pages to elementarily, recurrently, and inductively prove that the ordinary differential equations

$$F_{\mp}^{(N)} = \left(-\frac{1}{\lambda}\ln(1+\lambda)\right)^{N}\sum_{i=1}^{N+1} a_{i-1}^{\mp}(N)F_{\mp}^{i}, \quad N = 0, 1, 2, \ldots \tag{1}$$

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have a solution

$$F_{\mp} = F_{\mp}(t) = \frac{1}{(1+\lambda)^{t/\lambda} \mp 1},$$

where $a_0^{\mp}(N) = 1$ and

$$a_j^{\mp}(N) = (-1)^{(j\mp j)/2} j! \sum_{i_j=0}^{N-j} \sum_{i_{j-1}=0}^{N-j-i_j} \cdots \sum_{i_1=0}^{N-j-i_j-\cdots-i_2} (j+1)^{i_j} j^{i_{j-1}} \cdots 2^{i_1} \tag{2}$$

for $1 \le j \le N$. With the aid of the quantities $a_j^{\mp}(N)$ in (2), the authors further obtained in [1, Theorems 3 and 6, Corollaries 4, 5, and 7] several identities and explicit expressions of the modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the higher order modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the Euler numbers E_n, the higher order Euler numbers E_n, the modified degenerate Bernoulli numbers $\tilde{\beta}_n(\lambda)$, the higher order modified degenerate Bernoulli numbers $\tilde{\beta}_n^{(r)}(\lambda)$, the Bernoulli numbers B_n, and the higher order Bernoulli numbers $B_n^{(r)}$, which can be generated respectively by

$$\frac{2}{(1+\lambda)^{t/\lambda}+1} = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n(\lambda)\frac{t^n}{n!}, \qquad \left[\frac{2}{(1+\lambda)^{t/\lambda}+1}\right]^r = \sum_{n=0}^{\infty} \tilde{\mathcal{E}}_n^{(r)}(\lambda)\frac{t^n}{n!},$$

$$\frac{2}{e^t+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!}, \qquad \left(\frac{2}{e^t+1}\right)^r = \sum_{n=0}^{\infty} E_n^{(r)}\frac{t^n}{n!},$$

$$\frac{t}{(1+\lambda)^{t/\lambda}-1} = \sum_{n=0}^{\infty} \tilde{\beta}_n(\lambda)\frac{t^n}{n!}, \qquad \left[\frac{t}{(1+\lambda)^{t/\lambda}-1}\right]^r = \sum_{n=0}^{\infty} \tilde{\beta}_n^{(r)}(\lambda)\frac{t^n}{n!},$$

$$\frac{t}{e^t-1} = \sum_{n=0}^{\infty} B_n\frac{t^n}{n!}, \qquad \left(\frac{t}{e^t-1}\right)^r = \sum_{n=0}^{\infty} B_n^{(r)}\frac{t^n}{n!},$$

where $r \in \mathbb{N}$. This means that the quantities $a_j^{\mp}(N)$ in (2) play an important role in the paper [1].

It is clear that the quantities $a_j^{\mp}(N)$ in (2) are formulated by j multiple sums. To the best of our knowledge, one cannot understand and compute easily such a j multiple sum. Then we guess that, making use of some different methods from the one employed in the paper [1], the quantities $a_j^{\mp}(N)$ in (2) should be reformulated simply, meaningfully, and significantly in terms of some mathematical quantities.

1.2. Second motivation

In the paper [2], Theorem 1 reads that the function

$$F_q(t) = \frac{1}{qe^t+1}$$

satisfies

$$(N-1)!F_q^N = \sum_{k=1}^{N} a_k(N) F_q^{(k-1)}$$

for $N \ge 1$ and $q \in \mathbb{R}$, where

$$a_k(N) = (-1)^{N+k} s(N,k) \tag{3}$$

or

$$a_k(N) = \frac{N!}{k!} \sum_{\substack{\ell_1,\dots,\ell_k \ge 1 \\ \ell_1+\cdots+\ell_k=N}} \frac{1}{\ell_1+\cdots+\ell_k}, \tag{4}$$

where the Stirling numbers of the first kind $s(n,k)$ can be generated by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1.$$

The proof of [2, Theorem 1] strides across five pages.

A close, similar, and recursive form of (4) was also obtained in [3, Theorem 1] and [4, p. 745]. Please also refer to [5, Remark 1].

In [6, Corollary 2.3], it was inductively and recursively procured that the Stirling numbers of the first kind $s(n,k)$ for $1\le k\le n$ can be expressed as

$$s(n,k)=(-1)^{n+k}(n-1)!\sum_{\ell_1=1}^{n-1}\frac{1}{\ell_1}\sum_{\ell_2=1}^{\ell_1-1}\frac{1}{\ell_2}\cdots\sum_{\ell_{k-2}=1}^{\ell_{k-3}-1}\frac{1}{\ell_{k-2}}\sum_{\ell_{k-1}=1}^{\ell_{k-2}-1}\frac{1}{\ell_{k-1}}. \tag{5}$$

See also [7, p. 27, Remark 3.3], [8, p. 244, Eq. (2.1)], [9, p. 2308, Eq. (1.8)], [10, p. 6], the papers [11,12] and the closely related references therein. Comparing (4) with (5), we observe that

$$(-1)^{n+k}s(n,k)=\frac{n!}{k!}\sum_{\substack{\ell_1,\dots,\ell_k\ge1\\ \ell_1+\dots+\ell_k=n}}\frac{1}{\ell_1+\dots+\ell_k},\quad n\ge k\ge1.$$

See also [13, pp. 7–8, Section 2.4], [14, p. 9, Section 2.5], and [5, Eq. (5)]. Consequently, the two expressions (3) and (4) are essentially the same one.

1.3. Third motivation

The third motivation was stated in the paper [15] and in its preprint [16]. For shortening the length of current paper, we do not repeat it in details once again.

1.4. Main results

Our main results of this paper can be stated as the following theorem.

Theorem 1. *For $n\in\mathbb{N}$, the nonlinear ordinary differential equations*

$$F^{(n)}(t)=(-1)^n\left[\frac{\ln(1+\lambda)}{\lambda}\right]^n\sum_{k=1}^{n+1}(k-1)!S(n+1,k)F^k(t) \tag{6}$$

and

$$F^n(t)=\frac{1}{(n-1)!}\sum_{k=1}^{n}(-1)^{k-1}\left[\frac{\lambda}{\ln(1+\lambda)}\right]^{k-1}s(n,k)F^{(k-1)}(t) \tag{7}$$

have the same solutions

$$F(t)=\frac{1}{\beta(1+\lambda)^{t/\lambda}-1},$$

where $\beta\ne0$ and $S(n,k)$ are the Stirling numbers of the second kind which can be generated by

$$\frac{(e^x-1)^k}{k!}=\sum_{n=k}^{\infty}S(n,k)\frac{x^n}{n!},\quad k\in\mathbb{N}.$$

2. A lemma

In order to prove Theorem 1, we need the following lemma.

Lemma 1 (*[17, Theorem 2.1] and [18, Theorems 3.1 and 3.2]*). *Let $\alpha, \beta \ne 0$ be real constants and $k \in \mathbb{N}$. When $\beta > 0$ and $t \ne -\frac{\ln \beta}{\alpha}$ or when $\beta < 0$ and $t \in \mathbb{R}$, we have*

$$\frac{\mathrm{d}^k}{\mathrm{d}\, t^k}\left(\frac{1}{\beta e^{\alpha t}-1}\right) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1,m) \left(\frac{1}{\beta e^{\alpha t}-1}\right)^m \tag{8}$$

and

$$\left(\frac{1}{\beta e^{\alpha t}-1}\right)^k = \frac{1}{(k-1)!}\sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k,m) \frac{\mathrm{d}^{m-1}}{\mathrm{d}\, t^{m-1}}\left(\frac{1}{\beta e^{\alpha t}-1}\right). \tag{9}$$

Remark 1. For motivations and further developments about the papers [17,18], please refer to [19–21] and the closely related references therein.

3. Proof of Theorem 1

It is easy to see that

$$(1+\lambda)^{t/\lambda} = e^{t \ln(1+\lambda)^{1/\lambda}} = e^{\alpha t}, \quad \alpha = \frac{\ln(1+\lambda)}{\lambda}.$$

Consider the function

$$F_{\alpha,\beta}(t) = \frac{1}{\beta e^{\alpha t}-1}, \quad \beta \ne 0.$$

From (8) and (9), it follows that

$$\frac{\mathrm{d}^k}{\mathrm{d}\, t^k} F_{\alpha,\beta}(t) = (-1)^k \alpha^k \sum_{m=1}^{k+1} (m-1)! S(k+1,m) [F_{\alpha,\beta}(t)]^m$$

and

$$[F_{\alpha,\beta}(t)]^k = \frac{1}{(k-1)!}\sum_{m=1}^{k} \frac{(-1)^{m-1}}{\alpha^{m-1}} s(k,m) \frac{\mathrm{d}^{m-1}}{\mathrm{d}\, t^{m-1}} F_{\alpha,\beta}(t).$$

The proof of Theorem 1 is thus complete.

4. Remarks

Finally, we list several remarks to explain and understand some related results.

Remark 2. Comparing (1) with (6) reveals that

$$S(n+1,k) = \sum_{i_{k-1}=0}^{n-k+1} \sum_{i_{k-2}=0}^{n-k+1-i_{k-1}} \cdots \sum_{i_1=0}^{n-k+1-i_{k-1}-\cdots-i_2} k^{i_{k-1}} (k-1)^{i_{k-2}} \cdots 2^{i_1}. \tag{10}$$

Remark 3. Considering the relations (2) and (10), we can simply, significantly, and meaningfully restate identities and explicit expressions of the modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the higher order modified degenerate Euler numbers $\tilde{\mathcal{E}}_n(\lambda)$, the Euler numbers E_n, the higher order Euler numbers E_n, the modified degenerate Bernoulli numbers $\tilde{\beta}_n(\lambda)$, the higher order modified degenerate Bernoulli numbers $\tilde{\beta}_n^{(r)}(\lambda)$, the Bernoulli numbers B_n, and the higher order Bernoulli numbers $B_n^{(r)}$ in Theorem 3, Corollary 4, Corollary 5, Theorem 6, and Corollary 7 of the paper [1] in terms of the Stirling numbers of the second kind $S(n,k)$.

Remark 4. About simplifying the differential equations in (1) as those in (6), as well as (7), the idea originates from the papers [13,14,16,19,22–26] and the closely related references therein.

Remark 5. This paper is a slightly revised version of the preprint [27].

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