Systematization of basic divergent integrals in perturbation theory and renormalization group functions

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We show that to n loop order the divergent content of a Feynman amplitude is spanned by a set of basic (logarithmically divergent) integrals $I^{(i)}_{\log}(\lambda^2)$, $i = 1, 2, \ldots, n$, $\lambda$ being the renormalization group scale, which need not be evaluated. Only the coefficients of the basic divergent integrals are show to determine renormalization group functions. Relations between these coefficients of different loop orders are derived.

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\section{1. Introduction}

Implicit regularization (IR) is a non-dimensional momentum space framework which has been claimed to be a strong candidate for an invariant regularization suitable to develop perturbation theory in supersymmetric gauge field theories \cite{1–17}. Assuming an implicit regulator in a general (multiloop) Feynman amplitude, a mathematical identity at the level of propagators allows to write the divergent content as basic divergent integrals (BDI) or loop integrals written in terms of one internal momentum only in an unitarity preserving fashion. This is possible because BPHZ subtractions as well as the counterterm method are compatible with IR to arbitrary loop order. An arbitrary scale appears via a regularization-independent identity which relates two logarithmically BDIs by trading a mass parameter $m$ (or an infrared regulator in the propagators) for an arbitrary positive parameter $\lambda$ ($[\lambda] = M$) plus a function of $m/\lambda$. Consequently, $\lambda$ parametrizes the freedom of separating the divergent content of an amplitude and acts as a renormalization group scale. The key point underlying IR is that neither the (regularization-dependent) BDIs nor their derivatives with respect to $\lambda$ represented by BDIs need be evaluated. In other words, the BDIs are readily absorbed into renormalization constants whose derivatives with respect $\lambda$ used to calculate renormalization group functions can also be expressed by BDIs. The advantage of such scheme is that a physical amplitude is written as a finite part plus a set of BDIs say $I^{(i)}_{\log}(\lambda^2)$ and finite surface terms (STs) expressed by volume integrals of a total derivative in momentum space which stem from (finite) differences between $I^{(i)}_{\log}(\lambda^2)$ and $I^{(i)}_{\log}(\mu_1,\mu_2,\ldots)(\lambda^2)$ where the latter is a logarithmically divergent integral which contains in the integrand a product of internal momenta carrying Lorentz indices $\mu_1, \mu_2, \ldots$. In other words throughout the reduction of the amplitude to loop integrals, $I^{(i)}_{\log}(\mu_1,\mu_2,\ldots)(\lambda^2)$ may be written as a product of metric tensors symmetrized in the Lorentz indices times $I^{(i)}_{\log}(\lambda^2)$ plus a surface term.

Such STs are in principle arbitrarily valued. However, it has been shown that setting them to zero ab initio corresponds to both invoking translational invariance of Green's functions and allowing shifts in the integration variable in momentum space \cite{4,5} which in turn is an essential ingredient to demonstrate gauge invariance based on a diagrammatic proof. Therefore STs seem to encode the possible symmetry breakings. Moreover, it has been verified that constraining such surface terms to nought is also sufficient to guarantee that supersymmetry is preserved in the Wess–Zumino model to 3rd-loop order \cite{10} and supergravity to 1-loop order \cite{14}. Notwithstanding it is reasonable to assert that IR is a good candidate to an invariant calculational friendly regularization framework valid in arbitrary loop order. From

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the point of view of algebraic renormalization, STs would be the necessary symmetry restoring counterterms whose expression is known within IR. Then a constrained version of IR (CIR) amounts to setting them to zero from the start and thus constituting an invariant scheme. When physical quantum breakings (anomalies) are expected some care must be exercised: one is able to spot a genuine breaking by letting the STs to be arbitrary so to verify that none consistent set of values for the STs dictated by symmetry requirements fulfill all the essential Ward identities of the underlying model at the same time [7,12]. In [2,5] the rules that define IR to arbitrary loop order are specified.

A renormalization group equation can immediately be written within IR adopting $\lambda$ as a renormalization group scale and a minimal, mass-independent renormalization scheme in which only the basic divergent integrals are absorbed in the renormalization constants.

Hence renormalized Green’s function satisfy a kind of Callan–Symanzik equation governed by the scale $\lambda$.

The purpose of this contribution is to twofold. Firstly although IR works in arbitrary massive quantum field theories, for massless theories it undergoes a remarkable simplification. Assuming an infrared regulator $\mu$ for the propagators, $l_{\text{log}}(\mu^2)$ equals $l_{\text{log}}(\lambda^2)$ ($\lambda \neq 0$), plus a sum of terms proportional to powers of the logarithm of the ratio $\mu/\lambda$. We will show in this contribution that for massless theories all the divergencies to arbitrary loop order can be cast as a function of $l_{\text{log}}(\lambda^2)$, according to the definition

$$l_{\text{log}}(\mu^2) = \int_{k}^{\Lambda} \frac{1}{(k^2 - \mu^2)^2} \ln^{i-1}(\frac{k^2 - \mu^2}{\lambda^2}).$$

where $\int_{k}^{\Lambda} \equiv \int(d^4k)/(2\pi)^4$ and the superscript $\Lambda$ is a symbol for an implicit regularization. Secondly it is well known that renormalization group functions constitute a testing ground for regularizations because they both encode the symmetry properties of the underlying model which should be preserved by the regularizations and their expansion in perturbation theory contains terms which are universal, i.e. renormalization scheme-independent. While some interesting simplifications take place in dimensional methods, e.g. in an inverse power series in $\epsilon \rightarrow 0$ of the coupling constant, beta functions are determined uniquely by the residue of the simple pole on $\epsilon$, it is pertinent to ask what is the counterpart in IR. That is to say, one may wonder how the calculation of renormalization group functions systematizes within a scheme where only basic divergent integrals are claimed to be sufficient to exhibit the ultraviolet properties of a model in a symmetry preserving fashion. The answer to this question is that a general framework for renormalization group functions can be built in which the simplifications of dimensional methods manifest themselves as relations between the coefficients of basic divergent integrals coming from different Feynman graphs that contribute to a given renormalization group function.

We illustrate with the Yukawa model in $3 + 1$ dimensions to 2nd-loop order which contains a $\gamma_\mu$ matrix and hence the application of dimensional regularization is more involved.

2. General ultraviolet structure of massless theories

The purpose of this section is to show that the ultraviolet content of an amplitude to $n$th order for massless models, considering the definition, is written in terms of $l_{\text{log}}(\lambda^2)$. A general $n$-loop, $l$-point amplitude, after space–time and internal group algebra contractions are performed, can always be written as a combination of integrals of the type

$$\int_{k}^{\Lambda} \frac{k_{\mu_1}k_{\mu_2} \cdots k_{\mu_l}}{(k - p_1)^2 \cdots (k - p_l)^2} A_{n-1}(k, p_1, \ldots, p_l, \lambda^2),$$

where we have integrated $n - 1$ times leaving only $k$, the most external loop momentum and the $p_i$’s are external momenta. For a massless model suppose that $A_{n-1}$ is cast like

$$A_{n-1}(k, p_1, \ldots, p_l, \lambda^2) = A^A_{n-1} + \sum_{i=1}^{n} a_i(k, p_1, \ldots, p_l) \ln^{i-1}(\frac{k^2 - \mu^2}{\lambda^2}) + \tilde{A}_{n-1},$$

in which $\tilde{A}_{n-1}$ is finite under integration on $k$ and $A^A_{n-1}$, the divergent part, represents the subdivergences which in principle are already written in terms of $l_{\text{log}}(\lambda^2)$. The mass scale $\lambda^2$ has emerged from a scale relations which characterizes a renormalization scheme in implicit regularization. the coefficients $a_i(k, p_1, \ldots, p_l)$ may contain powers in the external and internal momenta. To justify the assumption of Eq. (3) we proceed with a proof by induction. For $n = 2$ (one loop order) it can be easily verified that (3) holds for $A_1$ [2]. Now we show that this assumption for $(n - 1)$th-loop order implies the same structure for the $n$th-loop order to conclude by induction that the multiloop integrals at any order have the same structure. The relevant contributions come from the second term on the r.h.s. of (3),

$$\int_{k}^{\Lambda} \frac{k_{\mu_1} \cdots k_{\mu_{n-1}}}{(k - p_1)^2 - \mu^2} \cdots (k - p_{n-1})^2 - \mu^2} \ln^{i-1}(\frac{k^2 - \mu^2}{\lambda^2}).$$

which has superficial degree of divergence $r(i) - 2l + 4$. Extra factors in the numerator were considered so as to account for the Lorentz structure of the $a_i(k, p_1, \ldots, p_l)$’s. A fictitious mass $\mu^2$ was introduced in the propagators and the limit $\mu^2 \rightarrow 0$ will be taken in the end. A fictitious mass may always be introduced if the integral is infrared safe. This is necessary because although the integral is infrared safe, the expansion of the integrand, as we explain below, breaks into infrared-divergent pieces. When a genuine infrared divergence appears, this procedure can be problematic in non-Abelian theories. For such cases a new procedure within IR defining basic infrared-divergent integrals is necessary in order to preserve symmetries [13].

We judiciously apply in the integrand the identity,

$$\frac{1}{(p_i - k)^2 - \mu^2} = \frac{1}{(k^2 - \mu^2)} - \frac{p_i^2 - 2p_i \cdot k}{(k^2 - \mu^2)(|p_i - k|^2 - \mu^2)}.$$
We still have to deal with the fictitious mass, which in the limit $\mu \to 0$ will give infrared-divergent pieces both in the ultraviolet-divergent and finite parts. This problem is simply dealt with by the use of regularization-independent scale relations (they can be easily obtained with the help of a cutoff), which read

$$l_{\log}^{(ij)\mu\nu}(\mu^2) = \frac{g^{ij \mu \nu}}{4} \sum_{i=1}^{j} \frac{(j-1)!}{(i-1)!} l^{(ij)}(\mu^2) + \text{surface terms.}$$

(11)

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(11)
\begin{align}
J_{\mu_1...\mu_n} &= \frac{1}{i} \int \frac{d^n k}{(2\pi)^n} \ln^{n-1} \left( -\frac{k^2}{\lambda^2} \right) - \int \frac{d^2 p}{(2\pi)^2} \left[ 2 \sum_{k=0}^{n-1} \left( \frac{-p^2}{k!} \right)^n \ln \left( -\frac{k^2}{\lambda^2} \right) \right],
\end{align}
and $\Gamma^{\phi^2} = iA\phi^2 + i\Gamma_{2h} + i\Gamma_{2c} + i\Gamma_{2d} + i\Gamma_{2e} + i\Gamma_{2f} + i\Gamma_{2g} + i\Gamma_{2h}$ which yields

$$\Gamma^{\phi^2} = iA\phi^2 + \frac{p^2}{16\pi^2} \left[ (32\pi^2 e^2 - 7e^4 + \frac{g^2}{12} I_{\log}(\lambda^2) - 32\pi^2 i e^6 I_{\log}(\lambda^2))^2 + 2e^4 I_{\log}^{(2)}(\lambda^2) \right].$$

(24)

In determining the counterterm graphs that correspond to the amplitudes in Eqs. (19) and (20) above we have used the one loop contributions of the counterterms $B$ and $C$ from Eq. (28). Notice that the non-local divergences have been correctly canceled as they should because we have shown in [2] that IR is compatible with the counterterm method derived from BPHZ forest formula. Diagrams in Figs. 3, 4 and 5 are evaluated in a similar fashion, using one loop counterterms previously determined, to give

$$\Gamma^{\phi \psi} = iB\phi^2 + \frac{p^2}{16\pi^2} \left[ \left( 8\pi^2 e^2 - \frac{31}{8} e^4 \right) I_{\log}(\lambda^2) + \frac{9}{4} e^4 I_{\log}^{(2)}(\lambda^2) \right],$$

(25)

$$\Gamma^{\phi \psi \phi} = -ieC\gamma_5 + \frac{ig\gamma_5}{16\pi^2} \left[ (-16\pi^2 e^3 + 9e^5) I_{\log}(\lambda^2) - 6e^5 I_{\log}^{(2)}(\lambda^2) \right],$$

(26)

$$\Gamma^{\phi \phi} = -igD + \left( 24i\pi^2 e^2 - 384\pi^2 e^4 - 6e^3 - 12g^2 e^2 + 336e^6 + 96ge^4 \right) \frac{I_{\log}(\lambda^2)}{16\pi^2}$$

$$+ (3g^3 + 6g^2 e^2 - 144e^6 - 72ge^4) \frac{I_{\log}^{(2)}(\lambda^2)}{16\pi^2} + i \left( -\frac{3}{4} g^3 - 96e^6 - 72ge^4 \right) \left[ I_{\log}(\lambda^2) \right]^2,$$

(27)

respectively.

Then we have the renormalization constants defined in a minimal scheme if

$$A = \frac{i}{16\pi^2} \left[ \left( \frac{g^2}{12} + 32\pi^2 e^2 \right) I_{\log}(\lambda^2) + e^4 \left( -7I_{\log}(\lambda^2) - 32i\pi^2 I_{\log}(\lambda^2)^2 \right) + 2I_{\log}^{(2)}(\lambda^2) \right].$$

$$B = \frac{i}{16\pi^2} \left[ e^2 8\pi^2 I_{\log}(\lambda^2) + e^4 \left( -\frac{31}{8} I_{\log}(\lambda^2) + \frac{9}{4} I_{\log}^{(2)}(\lambda^2) \right) \right].$$

$$C = \frac{i}{16\pi^2} \left[ -e^2 16\pi^2 I_{\log}(\lambda^2) + e^4 \left( 9I_{\log}(\lambda^2) - 6I_{\log}^{(2)}(\lambda^2) \right) \right].$$

$$D = \frac{i}{16\pi^2} \left[ g^2 24\pi^2 I_{\log}(\lambda^2) + g^2 \left( 6I_{\log}(\lambda^2) - 3I_{\log}^{(2)}(\lambda^2) - 12\pi^2 \left[ I_{\log}(\lambda^2)^2 \right] \right) + g^{-1} e^6 384\pi^2 I_{\log}(\lambda^2)$$

$$+ g^{-1} e^6 \left( -336I_{\log}(\lambda^2) + 144I_{\log}^{(2)}(\lambda^2) - 96(16\pi^2) I_{\log}(\lambda^2)^3 \right) + e^4 \left( -96I_{\log}(\lambda^2) + 72I_{\log}^{(2)}(\lambda^2) - 72(16\pi^2) I_{\log}(\lambda^2)^3 \right) \right].$$

(28)
The beta-functions and field anomalous dimensions are defined as usual
\[
\gamma_\phi = \frac{\lambda^2}{Z_\phi} \frac{\partial Z_\phi}{\partial \lambda^2}, \quad \gamma_\psi = \frac{\lambda^2}{Z_\psi} \frac{\partial Z_\psi}{\partial \lambda^2},
\]
\[
\rho_e = -g\lambda^2 \left( Z_e^{-1} \frac{\partial Z_e}{\partial \lambda^2} - \frac{1}{2} \lambda^{-1} \frac{\partial Z_\phi}{\partial \lambda^2} - Z_\psi^{-1} \frac{\partial Z_\psi}{\partial \lambda^2} \right), \quad \rho_g = -2g\lambda^2 \left( Z_g^{-1} \frac{\partial Z_g}{\partial \lambda^2} - 2Z_\psi^{-1} \frac{\partial Z_\psi}{\partial \lambda^2} \right).
\]  
(29)
where \( \lambda \) is the IR arbitrary scale which plays the role of renormalization group scale. To \( n \)-loop order, a general renormalization constant can be written as
\[
Z = 1 + \sum_{j=1}^n \left( g^p Z_e^{(j)} + e^q Z_\phi^{(j)} + \lambda^r Z_\psi^{(j)} + \lambda^s Z_g^{(j)} \right),
\]
in which \( p, q, r, s \) assume positive integer values in each \( n \). In a minimal, mass-independent renormalization scheme, the renormalization constants \( Z_e^{(j)}, Z_\phi^{(j)} \) and \( Z_g^{(j)} \) take the general form
\[
Z_e^{(j)} = \sum_{k=1}^j b_k^{(j)} [\log(\lambda^2)]^k + \sum_{k=2}^j b_k^{(j)} [\log(\lambda^2)]^{k-2}, \quad Z_\phi^{(j)} = \sum_{k=1}^j b_k^{(j)} [\log(\lambda^2)]^{k-1}, \quad Z_g^{(j)} = \sum_{k=1}^j b_k^{(j)} [\log(\lambda^2)]^{k-2}.
\]  
(30)
For the Yukawa model to two loop order we have
\[
Z_\phi = 1 + g^2 Z_a^{(2)} + \sum_{n=2}^\infty e^{2n} Z_b^{(n)}, \quad Z_\psi = 1 + \sum_{n=1}^\infty e^{2n} Z_c^{(n)}, \quad Z_g = 1 + e^4 Z_f^{(2)} + \sum_{n=1}^\infty e^{2n} Z_g^{(2)} + \sum_{n=1}^\infty e^{2n+1} Z_h^{(n)} + \sum_{n=1}^\infty e^{2n} Z_i^{(n)}.
\]  
(32)
Now plugging Eqs. (32) into (29) permits us to obtain the finite contributions to renormalization group functions to 1 and 2-loop order from
\[
\gamma_\phi^{(1)} = e^2 \lambda^2 \frac{\partial Z_e^{(1)}}{\partial \lambda^2}, \quad \gamma_\psi^{(1)} = e^2 \lambda^2 \frac{\partial Z_\phi^{(1)}}{\partial \lambda^2}, \quad \rho_e^{(1)} = 2e^2 \lambda^2 \left( 1 - \frac{1}{2} \lambda^{-2} \frac{\partial Z_\phi}{\partial \lambda^2} + \frac{1}{2} \lambda^{-1} \frac{\partial Z_\psi}{\partial \lambda^2} \right), \quad \rho_g^{(1)} = 4ge^2 \lambda^2 \frac{\partial Z_\psi^{(1)}}{\partial \lambda^2} - 2e^4 \lambda^2 \frac{\partial Z_g^{(1)}}{\partial \lambda^2} + 2g^2 \lambda^2 \frac{\partial Z_e^{(1)}}{\partial \lambda^2}.
\]  
(33)
and
\[
\gamma_\phi^{(2)} = \lambda^2 \left( e^2 \lambda^2 \frac{\partial \log(Z_\phi)}{\partial \lambda^2} + e^4 \frac{\partial Z_\phi^{(2)}}{\partial \lambda^2} \right), \quad \gamma_\psi^{(2)} = e^4 \lambda^2 \frac{\partial Z_\psi^{(2)}}{\partial \lambda^2}, \quad \rho_e^{(2)} = -2e^2 \lambda^2 \left( \frac{\partial Z_a^{(2)}}{\partial \lambda^2} + \frac{\partial Z_b^{(2)}}{\partial \lambda^2} \right) + \lambda^2 \left( \frac{\partial Z_a^{(2)}}{\partial \lambda^2} + \frac{\partial Z_b^{(2)}}{\partial \lambda^2} \right),
\]  
\[
\rho_g^{(2)} = 2ge^4 \lambda^2 \left( \frac{\partial Z_g^{(2)}}{\partial \lambda^2} - \frac{1}{2} \frac{\partial Z_\phi}{\partial \lambda^2} \right) + 2g^2 \lambda^2 \left( \frac{\partial Z_g^{(2)}}{\partial \lambda^2} - \frac{1}{2} \frac{\partial Z_\psi}{\partial \lambda^2} \right) - 2ge^4 \lambda^2 \frac{\partial Z_\psi^{(2)}}{\partial \lambda^2} + 2e^4 \lambda^2 \frac{\partial Z_g^{(2)}}{\partial \lambda^2}.
\]  
(34)
To complete our task we have to evaluate the derivatives of (31) w.r.t. \( \lambda^2 \) which are expressible in terms of BDIs as well, namely
\[
\lambda^2 \frac{\partial Z_a^{(n)}}{\partial \lambda^2} = \frac{\partial}{\partial \lambda^2} \left[ \left( \lambda^{2k-4} \right)^n \left( k! \right) \right], \quad \lambda^2 \frac{\partial Z_b^{(n)}}{\partial \lambda^2} = \frac{\partial}{\partial \lambda^2} \left[ \left( \lambda^{2k-4} \right)^n \left( k! \right) \right], \quad \lambda^2 \frac{\partial Z_c^{(n)}}{\partial \lambda^2} = \frac{\partial}{\partial \lambda^2} \left[ \left( \lambda^{2k-4} \right)^n \left( k! \right) \right].
\]
(35)
which agree with [18,19].

We can generalize (33) and (34) to arbitrary loop order using the expansion (30) in (29) to conclude that all we need to evaluate the renormalization functions is the derivative of \( Z^{(n)} \) as given in (35). It is interesting to remark that whilst the finite terms in the
r.h.s. of (35) contribute to the computation of the renormalization group functions, the terms proportional to BDIs will give relations between \(A^{(i)}(\lambda)\) and \(B^{(i)}(\lambda)\) as they must vanish because the renormalizations functions are finite. The same reasoning leads us to conclude, in dimensional regularization methods, that only residues of order one contribute to beta-functions. For instance, from the calculation of the field anomalous dimensions \(\gamma_\phi\) and \(\gamma_\lambda\) up to two loop order we get

\[
\begin{align*}
A^{(2)}_{a_{22}} - 8i\pi^2 B^{(2)}_{a_{22}} &= 0, \\
\langle A^{(2)}_{a_{22}} - 8i\pi^2 B^{(2)}_{a_{22}} \rangle &= \frac{1}{2} (A^{(1)}_{a_{22}})^2 + A^{(1)}_{a_{11}} \left( -\frac{1}{2} A^{(1)}_{b_{11}} + A^{(1)}_{e_{11}} - A^{(1)}_{c_{11}} \right), \\
\langle A^{(2)}_{a_{22}} - 8i\pi^2 B^{(2)}_{a_{22}} \rangle &= \frac{1}{2} (A^{(1)}_{c_{22}})^2 + A^{(1)}_{a_{11}} \left( -\frac{1}{2} A^{(1)}_{b_{11}} + A^{(1)}_{e_{11}} - A^{(1)}_{c_{11}} \right),
\end{align*}
\]

respectively.

To conclude, we have shown that in Implicit Regularization (IR), we can organize the divergent content of an amplitude to \(n\)th loop order in terms of a basis of basic divergent integrals (BDIs), namely \(\Gamma^{(i)}\), \(i = 1, \ldots, n\), where \(\lambda\) is the RG scale. The calculation of RG functions systematizes within IR for they can be written in terms of coefficients of BDIs. Such coefficients are shown to be inter-related which in turn allows us to restrict ourselves to a subset of BDIs at each loop order to evaluate RG functions.

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Appendix A

We calculate explicit the diagram \(g\) of Fig. 2:

\[
\Gamma_{g} = -\int_{\Lambda}^{\Lambda} \int_{\Lambda}^{\Lambda} \frac{(-g\gamma s)^i}{i} \left( \frac{i}{k} (-g\gamma s)^i \right) \frac{i}{k-p} \left( \frac{i}{(k-l)^2} \right).
\]

where \(l\) and \(k\) are internal momenta. Taking the trace of Dirac matrices and simplifying we obtain:

\[
\Gamma_{g} = -2ig^4 \int_{\Lambda}^{\Lambda} \frac{k^2 (p-l)^2 + l^2 (p-k)^2 - p^2 (l-k)^2}{l^2 k^2 (k-p)^2 (l-p)^2 (k-l)^2}.
\]

or

\[
\Gamma_{g} = -2ig^4 \left\{ -p^2 \int_{\Lambda}^{\Lambda} \frac{1}{k^2 (k-p)^2} \left( \frac{1}{l^2 (l-p)^2} + 2 \frac{1}{k^2} \left( \frac{1}{l} \right)^2 \right) \right\}.
\]

At this point we apply in each of these integrals the methods discussed in Section 2. After some algebra we get:

\[
\Gamma_{g} = \frac{g^4 p^2}{8\pi^2} \left\{ -5i\ln(\lambda^2) + i16\pi^2 \left[ i\ln(\lambda^2) \right]^2 + 2i\ln(\lambda^2) \ln \left( \frac{-p^2}{\lambda^2} \right) + \text{finite} \right\}.
\]

Observe that the third term on the r.h.s. of (A.4) is non-local and it must be canceled with the ones of the counterterm diagrams.

References