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# On the solutions of field equations due to rotating bodies in General Relativity

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**Abstract**

A metric, describing the field due to bodies in stationary rotation about their axes and compatible with a stationary electromagnetic field, has been studied in present paper. Using Lie symmetry reduction approach we have herein examined, under continuous groups of transformations, the invariance of field equations due to rotation in General Relativity, that are expressed in terms of coupled system of partial differential equations. We have exploited the symmetries of these equations to derive some ansätz leading to the reduction of variables, where the analytic solutions are easier to obtain by considering the optimal system of conjugacy inequivalent subgroups. Furthermore, some solutions are considered by using numerical methods due to complexity of reduced ordinary differential equations.



*Keywords:* General Relativity; Electromagnetic field; Lie symmetry method; Exact solutions.

## 1. Introduction

General Relativity describes phenomena on all scales in the Universe, from compact objects such as black holes, neutron stars, and supernovae to large-scale structure formations such as those involved in creating the distribution of clusters of galaxies. For many years, physicists, astrophysicists and mathematicians have striven to develop techniques for unlocking the secrets contained in Einstein's theory of gravity. More recently, solutions of Einstein field equations have added their expertise to the endeavor. Those who study these objects face a daunting challenge that the equations are among the most complicated in mathematical physics. Together, they form a set of coupled, nonlinear, hyperbolic-elliptic partial differential equations that contain many thousands of terms.

The gravitational field due to a rotating body was first attempted by Thirring who used Einstein field equations in the linear approximation and showed that a rotating thin spherical shell produces near its centre forces analogous to the Coriolis and centrifugal forces of classical machines. Later on this work has been revised by Pirani [23] who supplemented the energy tensor of incoherent material by a term representing the elastic interaction between the particles of the shell. Bach considered the field due to a slowly rotating sphere by successive approximations taking the Schwarzschild solution as his zeroth approximation. Special cases of stationary fields has been considered

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by Lanczos [16] and applied its results to cosmological problems.

Lewis [17] found the field due to a rotating infinite cylinder and thus obtained two different methods of successive approximations for constructing solutions of a more general type which behave in an assigned manner at infinity and on a surface of revolution enclosing the rotating matter to which the field is due. Clark tried to solve the empty gravitational field equations, using succussive approximations, with forms of $g_{\mu\nu}$ appropriate to the gravitational field of $s$ rotating body. This introduction provides a sample of the idea that these equations have been a subject of extensive and intensive study both by mathematicians and physicists. For the detail study of exact solutions of Einstein field equations, the reader may refer to Stephani et al. [24]. Recent years have been devoted to studying the field equations of General Relativity for their solutions [1–3,5–7,9,11,12,14,19,20], these solutions are important in the sense that they represent the physical models in analytic manner.

In the present paper, we have considered a metric [17] which is supposed to describe the field due to bodies in stationary rotation. Further in this case we furnished a consistent set of partial differential equations for determining $g_{\mu\nu}$ in empty space time. It is shown that by using the selective form of $g_{\mu\nu}$, the problem of solving four equations in three unknowns has been reduced to a system of two partial differential equations in two unknowns and then Lie group analysis is applied to generate the various symmetries of this coupled system of partial differential equations, which are then used to identify the associated basic vector fields of the optimal system for systematic study of the group invariant solutions admitted by the system.

## 2. Nature of field equations

The following metric described the field due to bodies in stationary rotation about their axes:

$$ds^2 = -\exp(2\lambda)(d\rho^2 + dz^2) - Cd\phi^2 + Ddt^2 + 2Ed\phi dt, \tag{2.1}$$

where $\lambda, C, D$ and $E$ are functions of $\rho$ and $z$ only.

Following Lewis [17], we have made use of canonical coordinates in the sense of Weyl. The choice of these coordinates is possible only in matter-free space as it can be easily be verified by a procedure similar to that of Synge. Consequently in domains occupied by matter the canonical coordinates cannot be used. In canonical coordinates we have

$$CD + E^2 = \rho^2, \tag{2.2}$$

and therefore the expressions for Einstein tensor are given by

$$G_{11} = -G_{22} = -\frac{\lambda_1}{\rho} - \frac{C_1D_1 + E_1^2 - C_2D_2 - E_2^2}{4\rho^2},$$

$$G_{33} = \frac{\exp(-2\lambda)}{2}\left(-2C(\lambda_{11}+\lambda_{22}) + C_{11} + C_{22} - \frac{C_1}{\rho} + \frac{3C}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\right),$$

$$G_{44} = \frac{\exp(-2\lambda)}{2}\left(2D(\lambda_{11}+\lambda_{22}) - D_{11} - D_{22} + \frac{D_1}{\rho} - \frac{3D}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\right),$$

$$G_{34} = \frac{\exp(-2\lambda)}{2}\left(-2E(\lambda_{11}+\lambda_{22}) - E_{11} - E_{22} + \frac{E_1}{\rho} - \frac{3E}{2\rho^2}(C_1D_1 + E_1^2 + C_2D_2 + E_2^2)\right),$$

$$G_{12} = -\frac{\lambda_2}{\rho} - \frac{C_1D_2 + 2E_1E_2 + C_2D_1}{4\rho^2}, \tag{2.3}$$

where lower suffixes 1 and 2 after the unknown functions imply partial differentiation with respect to $\rho$ and $z$ respectively.

Now we have considered the determination equation

$$|G_{\mu\nu} - sg_{\mu\nu}| = 0. \tag{2.4}$$

We found that two of the eigenvalues of $G_{\mu\nu}$ with respect to $g_{\mu\nu}$ are given by

$$s_i = \pm\exp(-2\lambda)\left(G_{22}^2 + G_{12}^2\right)^{\frac{1}{2}}, \qquad i = 1,\ 2, \tag{2.5}$$

and the other two are given by following equation

$$s^2 + Rs - \frac{1}{\rho^2}\left(G_{33}G_{44} - G_{34}^2\right) = 0, \tag{2.6}$$

where $R$ is curvature scalar. It is clear from Eqs. (2.5) and (2.6) that, in general, two eigenvalues of $G_{\mu\nu}$ are equal and opposite while other two are different. Therefore the metric (2.1) in canonical coordinates cannot represent a perfect fluid distribution. But if we do not consider the canonical coordinates then all the eigenvalues of Einstein tensor are different in general. Thus in this case metric (2.1) can be utilized to describe the space-time in the domains occupied by matter.

In case of an electromagnetic field, we have $R = 0$, therefore (2.6) gives

$$S_j = \pm\frac{1}{\rho}\sqrt{(G_{33}G_{44} - G_{34}^2)}, \quad j = 3, 4. \tag{2.7}$$

Thus in this case the other two eigenvalues are equal and opposite. Infact the eigenvalues are $k, -k, m, -m$, where

$$k = \exp(-2\lambda)\sqrt{(G_{22}^2 + G_{12}^2)},$$
$$m = \frac{1}{\rho}\sqrt{(G_{33}G_{44} - G_{34}^2)}, \tag{2.8}$$

and if we further consider $k = m$, the eigenvalues become $k, k, -k, -k$, which characterize an electromagnetic field.

### 2.1. The field equations for empty spacetime

The field equations, in terms of the coupled system of partial differential equations, for empty spacetime, corresponding to (2.1), are given by

$$\lambda_{11} + \lambda_{22} - \frac{\lambda_1}{\rho} = \frac{C_1D_1 + E_1^2}{2\rho^2}, \tag{2.9}$$

$$\lambda_{11} + \lambda_{22} + \frac{\lambda_1}{\rho} = \frac{C_2D_2 + E_2^2}{2\rho^2}, \tag{2.10}$$

$$\lambda_2 = -\frac{1}{4\rho}(C_1D_2 + C_2D_1 + 2E_1E_2), \tag{2.11}$$

$$C_{11} + C_{22} - \frac{C_1}{\rho} + \frac{C}{\rho^2}\left(C_1D_1 + C_2D_2 + E_1^2 + E_2^2\right) = 0, \tag{2.12}$$

$$D_{11} + D_{22} - \frac{D_1}{\rho} + \frac{D}{\rho^2}\left(C_1D_1 + C_2D_2 + E_1^2 + E_2^2\right) = 0, \tag{2.13}$$

$$E_{11} + E_{22} - \frac{E_1}{\rho} + \frac{E}{\rho^2}\left(C_1D_1 + C_2D_2 + E_1^2 + E_2^2\right) = 0. \tag{2.14}$$

From Eqs. (2.9) and (2.10), the condition of integrability can be easily verified for the above system of partial differential equations. Also from (2.9) and (2.10), we got

$$\lambda_{11} + \lambda_{22} = \frac{\left(C_1D_1 + C_2D_2 + E_1^2 + E_2^2\right)}{\rho^2}, \tag{2.15}$$

$$\lambda_1 = \frac{\left(-C_1D_1 + C_2D_2 - E_1^2 + E_2^2\right)}{4\rho}. \tag{2.16}$$

Also (2.15) is consistent with (2.11) and (2.16). Thus the problem of solving Eqs. (2.2), (2.9) to (2.14) reduces to determining $C$, $D$ and $E$ from (2.2), (2.12), (2.13) and (2.14) and then $\lambda$ will be given by (2.11) and (2.16).

We made the substitutions as follows:

$$C = \rho\exp(-\mu)\cos\theta, \quad D = \rho\exp(\mu)\cos\theta,$$
$$E = \rho\sin\theta, \tag{2.17}$$

where $\mu$ and $\theta$ are functions of $\rho$ and $z$. Consequently, (2.12), (2.13) and (2.14) is reduced to

$$\cos\theta\left(\mu_{11} + \mu_{22} + \tfrac{\mu_1}{\rho} - 2\tan\theta(\mu_1\theta_1 + \mu_2\theta_2)\right)$$
$$+\sin\theta\left(\theta_{11} + \theta_{22} + \tfrac{\theta_1}{\rho} - 2\sin\theta\cos\theta(\mu_1^2 + \mu_2^2)\right) = 0, \tag{2.18}$$

$$\cos\theta\left(\mu_{11} + \mu_{22} + \tfrac{\mu_1}{\rho} - 2\tan\theta(\mu_1\theta 1 + \mu_2\theta_2)\right)$$
$$-\sin\theta(\theta_{11} + \theta_{22} + \tfrac{\theta_1}{\rho} - 2\sin\theta\cos\theta(\mu_1^2 + \mu_2^2)) = 0, \tag{2.19}$$

$$\theta_{11} + \theta_{22} + \frac{\theta_1}{\rho} - \sin\theta\cos\theta(\mu_1^2 + \mu_2^2) = 0. \tag{2.20}$$

From (2.18) and (2.19) we got, in view of (2.20), the single equation:

$$\mu_{11} + \mu_{22} + \frac{\mu_1}{\rho} - \tan\theta(\mu_1\theta_1 + \mu_2\theta_2) = 0, \tag{2.21}$$

and (2.11) and (2.16) result into following equations:

$$\lambda_1 = -\frac{1}{4\rho} - \frac{\rho}{4}\left(\theta_1^2 + \theta_2^2 - \cos^2\theta(\mu_1^2 - \mu_2^2)\right), \tag{2.22}$$

$$\lambda_2 = \frac{\rho}{2}\left(\cos^2\theta\mu_1\mu_2 - \theta_1\theta_2\right). \tag{2.23}$$

Thus, the problem of solving four equations in three unknowns has been reduced to the system of partial differential equations consisting of two Eqs. (2.20) and (2.21) in two unknowns $\theta$ and $\mu$. Also we can determine $C, D$ and $E$ by using the expressions of $C, D$ and $E$ in (2.17) and then $\lambda$ can be determined from (2.22) and (2.23).

## 3. Solutions of field equations

It is well known that the Lie symmetries, originally advocated by the Norwegian mathematician Sophus Lie in the beginning of the 19th century, are widely applied to investigate nonlinear differential equations for constructing their exact and explicit solutions. Considering the tangent structural equations under one or several parameter transformation groups is the basic idea of the Lie symmetry analysis. It has been shown that the Lie symmetry analysis has been effectively used to look for exact and explicit solutions to both ordinary differential equations (ODEs) and partial differential equations (PDEs). There are a lot of papers and many excellent books [4,6,8,10,12,13,15,18,21,22] devoted to such applications.

In the present section, we have performed Lie group classification of Eqs. (2.20) and (2.21). That is, we have furnished all the possible forms of Lie point symmetries, admitted by Eqs. (2.20) and (2.21), and then constructed symmetry reductions and group-invariant solutions using the optimal system of subalgebras of the Lie algebras of the equations.

The classical Lie method [4] has been applied to Eqs. (2.20) and (2.21) by considering the one-parameter Lie group of infinitesimal transformations in $\rho, z, \theta, \mu,\ \xi^1(\rho,z), \xi^2(\rho,z), \eta^1(\rho,z)$ and $\eta^2(\rho,z)$. This transformation leaves invariant the following set:

$$S_\Delta \equiv \{\theta(\rho,\ z),\ \mu(\rho,\ z) : \Delta_1(\theta,\ \mu) = 0,\quad \Delta_2(\theta,\ \mu) = 0\}, \tag{3.1}$$

of solutions of Eqs. (2.20) and (2.21), where

$$\Delta_1 = \theta_{11} + \theta_{22} + \frac{\theta_1}{\rho} - \sin\theta\cos\theta(\mu_1^2 + \mu_2^2),$$
$$\Delta_2 = \mu_{11} + \mu_{22} + \frac{\mu_1}{\rho} - \tan\theta(\mu_1\theta_1 + \mu_2\theta_2). \tag{3.2}$$

The associated Lie algebra of infinitesimal symmetries is the set of vector fields of the form

$$\Gamma \equiv \xi^1\frac{\partial}{\partial\rho} + \xi^2\frac{\partial}{\partial z} + \eta^1\frac{\partial}{\partial\theta} + \eta^2\frac{\partial}{\partial\mu}. \tag{3.3}$$

The set $S_\Delta$ is invariant under the one-parameter transformations provided that $Pr^{(2)}(\Gamma)|_{\Delta=0} = 0$, where $Pr^{(2)}(\Gamma)$ is the second prolongation of the vector field $\Gamma$, which is explicitly given in terms of $\xi^1, \xi^2, \eta^1$ and $\eta^2$. After determining the infinitesimals of Eqs. (2.20) and (2.21), the similarity variables are derived by solving invariant surface conditions

$$\Phi_1 \equiv \xi^1\theta_\rho + \xi^2\mu_z - \eta^1 = 0,$$
$$\Phi_2 \equiv \xi^1\theta_\rho + \xi^2\mu_z - \eta^2 = 0. \tag{3.4}$$

The symmetries under which Eqs. (2.20) and (2.21) are invariant can be spanned by the following three linearly independent infinitesimal generators:

$$\Gamma_1 = \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z},\quad \Gamma_2 = \frac{\partial}{\partial z},\quad \Gamma_3 = \frac{\partial}{\partial\theta}. \tag{3.5}$$

It is easy to verify that $\Gamma_1, \Gamma_2$ and $\Gamma_3$ are closed under the Lie bracket. So we can see that the generator of invariant group $\Gamma$ of Eqs. (2.20) and (2.21) construct three-dimensional Lie algebra, which is spanned by the basis $\Gamma_1, \Gamma_2$ and $\Gamma_3$. Thus, we have the corresponding one-parameter group of symmetries of Eqs. (2.20) and (2.21):

$$G_1 : (\rho, z, \theta, \mu) \to (\exp(\epsilon)\rho,\ \exp(\epsilon)z,\ \theta,\ \mu),$$
$$G_2 : (\rho, z, \theta, \mu) \to (\rho,\ \exp(\epsilon)z,\ \theta,\ \mu),$$
$$G_3 : (\rho, z, \theta, \mu) \to (\rho, z,\ \exp(\epsilon)\theta,\ \mu). \tag{3.6}$$

We can see that $G_1$ is a space translation, $G_2$ is a time translation and $G_3$ is a scaling transformation. We have used the subalgebraic structure of symmetries (3.5) to construct an optimal system [22] of one dimensional subgroups. The optimal system yields only the following symmetry combinations:

$$\text{(i)}\ \Gamma_1 + \beta\Gamma_3,\quad \text{(ii)}\ \Gamma_2 + \alpha\Gamma_3,\quad \text{(iii)}\ \Gamma_3, \tag{3.7}$$

where $\alpha$ and $\beta$ are arbitrary constants.

### *3.1. Symmetry reductions*

In this subsection, we have derived symmetry reductions of Eqs. (2.20) and (2.21) associated with the vector fields in the optimal system (3.5) by using similarity variables and further attempted to furnish exact solutions.

**(i)** $\Gamma_1 + \beta\Gamma_3$

Corresponding to this vector field, the form of the similarity variable and similarity solution are as follows: $\zeta = \frac{\rho}{z},\ \theta(\rho,z) = F(\zeta),\ \mu(\rho,z) = \beta\log z + G(\zeta)$.

Substituting the expressions of the similarity variable and the similarity solution into Eqs. (2.20) and (2.21) yields the following system of reduced ODE:

$$\begin{aligned}&\zeta^3F'' + F' + \zeta + 2F'\zeta^2 - \zeta\beta^2\sin F\cos F\\ &\quad - 2\sin F\cos F\zeta^2\beta G' - \sin F\cos F\zeta^3G''^2\\ &\quad - \sin F\cos F\zeta G'^2 = 0,\quad \zeta G'' + G' + \beta\zeta\\ &\quad - G''\zeta^2 - 2\zeta^2G' - 2\zeta F'G'\tan F\\ &\quad + 2\zeta^2\beta F'\tan F - 2\zeta^3F'G'\tan F = 0.\end{aligned} \tag{3.8}$$

In this case because of the complexity of the reduced system (3.8), the following two particular cases have been worked out.

**Case (I):** By considering $F(\zeta) = 0$, we found that metric (2.1) is reduced to static axially symmetric metric of Weyl in canonical co-ordinates and system (3.8) becomes

$$\zeta G'' + G' + \beta\zeta - \zeta^2G'' - 2\zeta^2G' = 0. \tag{3.9}$$

Solving (3.9), we obtained the solutions as follows:

$$G(\zeta) = \frac{\beta}{4}\ln(-2\zeta) - \frac{\beta}{4}\ln(-2\zeta+2) + \frac{\beta}{2}\ln(\zeta-1) + c_1\mathrm{E}i(1,2\zeta) - c_1\exp(-2)\mathrm{E}i(1,2\zeta-2) + c_2, \tag{3.10}$$

where $c_1$ and $c_2$ are arbitrary constants and E$i$ is exponential integral. Now, we have obtained the solution of Eqs. (2.20) and (2.21) for static axially symmetric metric and further by back substitution to original variables, the exact solution of Eqs. (2.20) and (2.21) is given by:

$$\mu(\rho,z) = \frac{\beta}{4}\ln\left(\frac{-2\rho}{z}\right) - \frac{\beta}{4}\ln\left(\frac{-2\rho}{z}+2\right) + \frac{\beta}{2}\ln\left(\frac{\rho}{z}-1\right) + c_1\left(\mathrm{E}i\left(1,\frac{2\rho}{z}\right)\right) - c_1\left(\exp(-2)\mathrm{E}i\left(1,\frac{2\rho}{z}-2\right)\right) + c_2. \tag{3.11}$$

**Case (II):** By putting $G(\zeta)=0$, metric (2.1) is reduced to

$$ds^2 = -\exp(2\lambda)(d\rho^2+dz^2) - \rho\cos\theta(d\phi^2-dt^2) + 2\rho\sin\theta d\phi dt, \tag{3.12}$$

and then solving Eq. (3.8) and reverting back to the original variables. Thus we got the following exact solution of Eqs. (2.20) and (2.21):

$$\theta(\rho,z) = c_3 + \left(-\arctan\left(\frac{1}{\sqrt{\left(1+\frac{\rho^2}{z^2}\right)}}\right) + \left(\frac{(4+(\frac{\rho}{z})^2)}{3}\sqrt{\left(1+\frac{\rho^2}{z^2}\right)}\right)\right)c_4, \tag{3.13}$$

where $c_3$ and $c_4$ are arbitrary constants.

**(ii)** $\Gamma_2 + \alpha\Gamma_3$

For this vector field, the form of the similarity variable and similarity solution are as follows: $\zeta=\rho$, $\theta(\rho,z)=F(\zeta)$, $\mu(\rho,z)=G(\zeta)+\gamma z$. On using these in Eqs. (2.20) and (2.21), the system of reduced ODEs:

$$\zeta F'' + F' - \zeta\sin F\cos F(\alpha^2+G'^2) = 0,$$
$$\zeta G'' + G' - 2\zeta\tan F G = 0, \tag{3.14}$$

where prime (′) denotes the differentiation with respect to the variable $\zeta$.

Now under this vector field, we are unable to obtain the nontrivial exact solutions. So we have used a well-developed numerical technique to solve the reduced problem. For this purpose, we have obtained the following four first-order equations:

$$\frac{dy_1}{dz} = y_2,$$
$$\frac{dy_2}{dz} = \frac{-y_2+\alpha^2 z\sin y_1\cos y_1 + z\alpha^2 y_4^2\sin y_1\cos y_1}{z},$$
$$\frac{dy_2}{dz} = y_3, \quad \frac{dy_4}{dz} = \frac{-y_4+2y_2y_4 z\tan y_1}{z}, \tag{3.15}$$

with

$$y_1(45)=1.2, \quad y_2(45)=0, \quad y_3(45)=0,$$
$$y_5(45)=0.1. \tag{3.16}$$

The numerical solutions to the initial value problem (IVP) (3.15) and (3.16) are depicted below.

In Fig. 3, numerical solutions of field equations (2.20) and (2.21) are obtained with respect to the reduced IVP (3.15) and (3.16). Now the profile of $y1$ and $y2$ shows that the solution is periodic and the profile of $y3$ and $y4$ shows that the solution is unbounded and damped oscillatory respectively.

**(iii)** $\Gamma_3$

Corresponding to this vector field, no such invariant solution exists.

## 4. Discussion and concluding remarks

In the present investigation, we have successfully implemented Lie symmetry reduction to obtain the Lie

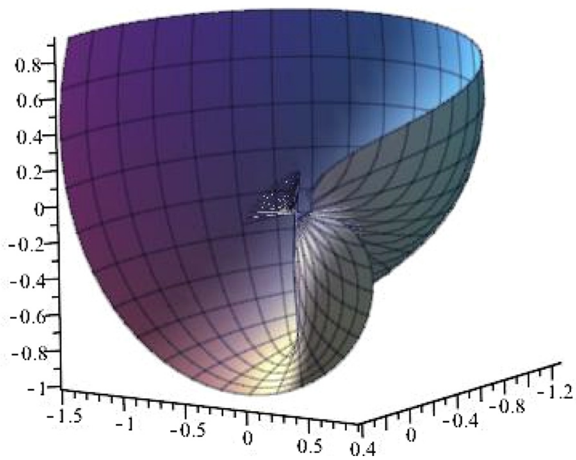


Fig. 1. Coriolis and centrifugal force solution (3.11), produced by rotating spherical shell in General Relativity to field (2.20) and (2.21) with $\beta=1, c_1=1$ and $c_2=1$.

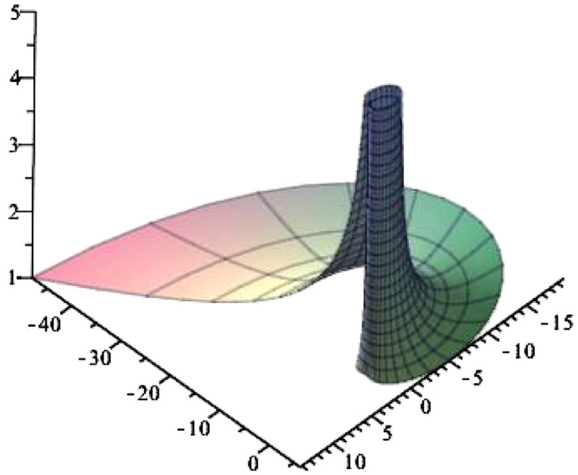


Fig. 2. Coriolis and centrifugal force solution (3.13), produced by rotating spherical shell in General Relativity to field (2.20) and (2.21) with $c_3 = 1$ and $c_4 = 1$.

symmetries admitted by field equations extracted from a metric which is supposed to describe the fields due to bodies in stationary rotation about their axes. The infinitesimal generators in the optimal system of sub algebras of the full Lie algebra of the coupled system of nonlinear partial differential equations of second order of field equations are considered. We completely solved the determining equations for the infinitesimal generators of Lie groups. Further, the group classification from the point of view of the optimal system of non-conjugate sub-algebras of the symmetry algebra of the nonlinear system has been performed under the adjoint action of the symmetry group. The various fields in the optimal system have been then exploited to get the reductions of PDEs into ODEs. Due to the complexity of reduced ODEs, it is impossible to obtain the nontrivial exact solutions, so under the vector field (i) $\Lambda_1 + \beta\Lambda_2$, particular exact solutions are obtained for the field equations (2.20) and (2.21). Graphical representation of solutions (3.11) and (3.13) to field

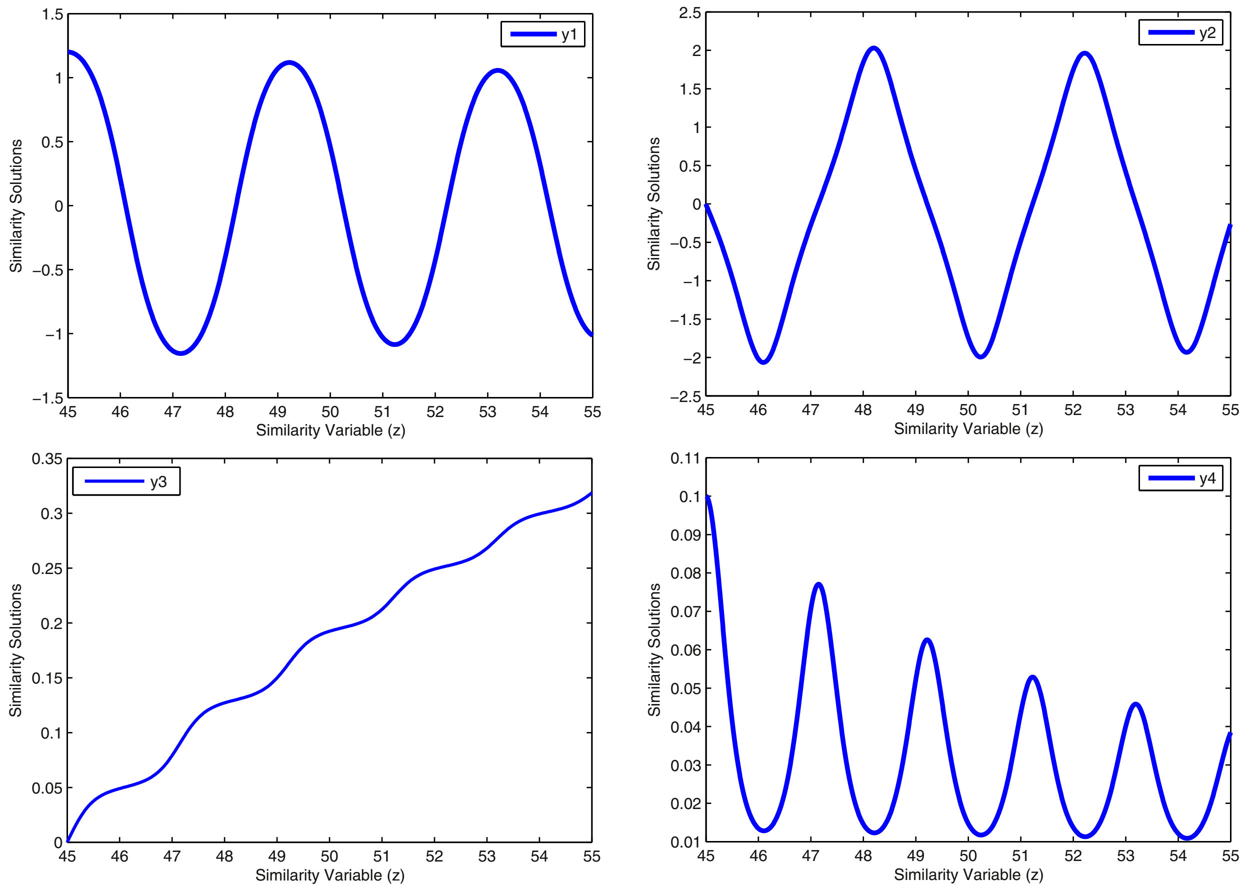


Fig. 3. Numerical solutions to field (2.20) and (2.21) with respect to the reduced IVP (3.15) and (3.16) when $\alpha^2 = -5$, with initial value $z = 45$ at $h = 0.01$.

equations (2.20) and (2.21) described the centre forces, similar to the Coriolis and centrifugal forces of classical machines, produced by the rotating spherical shell in General Relativity as shown in Figs. 1 and 2. Now under the vector field (ii) $\Lambda_2 + \alpha\Lambda_3$, it is again impossible to obtain the nontrivial exact solutions with respect to the reduced ODEs (3.14). So, under this vector field, IVP is posed for numerical solution. As a result, a numerical solution is found which is periodic, unbounded and damped oscillatory as shown in Fig. 3.

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